A Solution of the Clock Paradox

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In connection with Prof. Dingle's arguments that there is no asymmetry in the case of the «clock paradox» we show the difference between relativity and symmetry. Then a complete solution of the problem by the Special Relativity is given. Many opponent views are discussed and their errors are found. In the solution of the problem by the General Relativity an abridged exposition of Möller's calculations is given. Finally the experimental verification of the asymmetric ageing of mesons is discussed.

1. THE PROBLEM

One of the most discussed and controversial subjects of the Theory of Relativity is the so called «clock paradox». This problem may be stated in its usual form as follows : Suppose that we have two identical clocks A and B near each other, showing the same time. B moves away from A until it reaches a third point C and then it returns near A. It is known that, according to the Special Theory of Relativity B's time will go slower, as measured by A, while it moves away or to the clock A. «Therefore» it will show a smaller time when it will return near A. But, as the motion between A and B is relative, we may regard B at rest and A moving with exactly the opposite velocity of B at every moment. Then, according to the previous argument, A will be younger than B. This constitutes the paradox. What of the two descriptions is right? Or rather will A and B give the same reading at their meeting and no one will be younger than the other?

The above problem is not a new one. Einstein already in his original paper on the Special Theory of Relativity (**a**, p. 904) has stated that a clock moving in a circular orbit will show on its return a retardation. A nice illustration of this retardation was given by Langevin's (1911) «voyageur» who returns after a long travel much younger than his brother that stays at home. Eddington (**a**, p. 241; **b**, p. 28) gives a «justification» of the retardation of a moving clock based on the fact that only this clock suffers a real reversal of his motion, a disturbance, which is not relative, but absolute (see also M. v. Laue, **a**; **b**, p. 59). He makes a distinction between acceleration which is relative, and disturbance, which is not; however this distinction is not quite clear.

A proof of the retardation of the moving clock was given in 1934 by Tolman (p. 192 - 197). His proof is based on the General Relativity, but it is only approximate. The discussion continued after some years (Dingle **a**, **b**, **c**, Cambell) but no final conclusion was reached.

It seems a little strange that the problem of the clock paradox is not definitely settled as yet. The discussion continues even to day, more than 50 years after Einstein's original paper, and many people are not quite sure if the theory of Relativity implies or not an asymmetry between two similar clocks and whether travelling has any effect on the ageing of one person relative to another.

A few years ago McCrea (**a**) was writing that «it has never been made quite clear that there really is no (clock) paradox». The discussion that followed this paper, which we shall see presently, has proved that indeed the problem in not quite clear yet.

The general interest on the clock paradox has been revived these last years on account of the possibilities now arising of space travel. In some popular books on this subject the suggestion is made that space - travel may involve a lengthening of the life of the space-travellers. Of course the velocities likely to be attained in interplanetary flights are quite insufficient to give any significant time - dilatation; however the perspective of a lengthening of one's life is so much attractive! This is perhaps one reason why the problem of the time dilatation has become so popular to-day.

A long exchange of opposing views on this controversial subject has taken place these last two years. We mention mainly the papers that followed an exchange of views on the clock paradox between Professor Dingle and Professor McCrea in 1956-1957 (Dingle **d, e, f, g, i, j, k, l, m, n, o, p, q**), McCrea (**b, c, d, e, f, g, h**), Crawford (**a, b**), Singer, Cochran, Fisher, Halsbury, Weston, Builder (**a, b, c**), Fremlin, Darvin, Cullwick etc.). Of course the discussion is confined to the theoretical problem only, and especially in clarifying the implications of the theory of Relativity on the problem of the clock paradox. On the one side are Professor Dingle, Professor Cullwick and others. Dingle maintains in a very clear way, it must be admitted, the view of the exact symmetry of the motion of the clocks, which, he says, is an immediate consequence of the theory of Relativity. It is strange that although a great deal of authors have taken an opposite view, no one succeded in convincing him that he is wrong. In fact, if we except two replies, one by McCrea (**h**) and another by Crawford (**b**) which give a reasonable reply to Dingle's arguments, no one replied satisfactory to Dingle's basic argumentation, which is set forth in the form of a syllogism (**k, l**) :

1) According to relativity, if two bodies (e. g. two identical clocks) separate and reunite, there is no observable phenomenon that will enable one to say, in an absolute sense, that one rather than the other has moved.

2) If, on reunion, one clock is retarded by a quantity depending on the motion and the other is not, that phenomenon would enable one to say absolutely that the first had moved and not the second.

3) Hence, if relativity is true, the clocks must be retarded equally or not at all : in either case their readings will agree on reunion if they agreed at separation.

In this paper we shall point out first the weak point of this syllogism and show the basic difference between relativity and symmetry. Further we shall discuss the clock paradox by the Special and, separately, by the General Relativity, pointing out some errors of Dingle and others in this connection. Finally we shall refer to the experimental verification of the asymmetric ageing described by Crawford and give a reply to Dingle's objections to this.

2. RELATIVITY VERSUS SYMMETRY

It is evident that the motion of the two clocks in the case of the clock paradox is r e l a t i v e, i. e. we may regard either A moving relative to B, or B moving relative to A ; there is no absolute motion of course. However we must distinguish between t h e r e l a t i ve motion of the two bodies i r r e s p e c t i v e o f t h e r e s t o f t h e w o r l d, and t h e r e l a t i v e m o t i o n o f o n e b o d y w i t h r e s p e c t t o t h e r e s t o f t h e w o r l d.

If the only objects existing in the world were the two clocks of the clock paradox problem, then evidently there should be exact symmetry between them and Dingle should be right. In fact any force separating them should act symmetrically to them and any cause that should make these clocks to reunite (e.g. if the two clocks are connected by an elastic string which contracts after it is streched) it should again act symmetrically ; no case of asymmetry should arise.

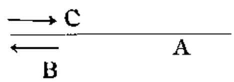
But in the real world there exist more than two objects of course. Therefore it is quite possible that the motion of the one clock with respect to the rest of the world is not symmetric to the motion of the other clock with respect to the corresponding rest of the world. We shall illustrate this by a simple example.

Suppose that there are three objects in the world on a straight line, say ABC, and that A and C are connected in such a way that no change of distance between them is possible. The «moving» clock B goes from A to C, where its motion is reversed by a collision and it returns to A. B is moving with respect to t h e s y s t e m AC; therefore we may regard B at rest and t h e s y s t e m AC m o v i n g with respect to B. This motion is exactly symmetric to the previous motion of B with respect to AC, i.e. B is symmetric to AC as regards their motions. But AC is not anything similar to B. It is composed of a clock A and a third object C. Therefore no conclusion about the equality of the readings of the clocks A and B can be drawn. It is seen on the contrary that A and B are not symmetric in their motion with respect to the rest of the world. Namely A's motion with respect to BC is not at all symmetric with B's motion with respect to AC. Therefore there is no apriori reason, based on the theory of Relativity, that the two clocks A and B should show the same readings on their reunion. As we shall see further, it is possible to prove, by the General Theory of Relativity, that B and not A will be retarded. In the following paragraphs we shall show how some specific «proofs» of Dingle and Cullwick about the equality of A's and B's readings on their reunion are invalidated. For the present we notice that Dingle's postulate of symmetry can be p r o v e d in general false, when three or more bodies exist in the world.

In a recent reply to Crawford (**b**) Dingle (**k**) denies the asymmetry because the assumed asymmetric ageing is «a quantity depending on their relative motions» namely $2\,T\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)$. This however is no objection at all to our syllogism. The asymetric ageing is a function of v, where v is the velocity of B with respect to AC, or the velocity of AC with respect to B. This shows that we cannot say if B or AC has moved in any absolute sense. But this does not change the fact that B's motion is not symmetric to A's. In fact *the asymmetric ageing is a function of the velocities of A and B with respect to the rest of the world* (here the point C).

3. THE CLOCK PARADOX IN SPECIAL RELATIVITY

In the case of the Special Theory of Relativity the clock paradox may be formulated as follows (Ives 1951):



Two clocks A and B are moviug with respect to each other with uniform velocity v. At some instant t=0 the two clocks go by each other and are then synchronized. At a later moment B meets a third clock C approaching A with velocity v; then B and C are synchronized. When C will meet A, will it show a smaller reading than A or not? It is evident that the motion of the system BC with respect to A is not symmetric to the motion of AC with respect to B. The corresponding symmetric experiment, when B is regarded at rest, should involve a fourth clock D moving toward B with velocity v and meeting A when A shows the same time that shows B when it meets C; theu D will be slow with respect to B, on meeting it, by the same amount as C is slow with respect to A when it meets it. It is seen in this experiment that an asymmetry does not want necessarily an acceleration to occur. However in the case of a *returning* clock an acceleration is necessary, and a «third boby acceleration», as Crawford (**b**) calls it, introduces the asymmetry in the problem.

The clock paradox in the case of the Special Relativity has been solved in first approximation by Ives. We shall give a complete solution here: When B and C meet, their times are $t_B = t_C = t_A \sqrt{1-\frac{v^2}{c^2}}$ where t_A is the reading of a clock at rest with respect to A and synchronous to it at the point of B's and C's meeting. When C and A meet, C's time is $\overline{t}_C = t_C + (\overline{t}_A - t_A)\sqrt{1-\frac{v^2}{c^2}} = \overline{t}_A\sqrt{1-\frac{v^2}{c^2}}$. If now we regard B at rest, then A and C are moving in the same direction. C's velocity with respect to A is v, i.e. its velocity with respect to B is $u = \frac{2v}{1+\frac{v^2}{c^2}}$. When B and C meet, their readings are $t_B' = t_C'$.

When C and A meet, their readings according to B are:

$\overline{t}_C' = t_B' + (\overline{t}_B' - t_B')\sqrt{1-\frac{u^2}{c^2}}$ and $\overline{t}_A' = \overline{t}_B'\sqrt{1-\frac{v^2}{c^2}}$, i.e. both run slower than B. Further: $v\,\overline{t}_B' = (\overline{t}_B' - t_B')u$, hence:

$t_B' = \frac{\overline{t}_B'}{2}\left(1-\frac{v^2}{c^2}\right)$. It is also $\sqrt{1-\frac{u^2}{c^2}} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{v^2}{c^2}\right)}$. Therefore:

$$\overline{t}_C' = \overline{t}_B'\left(1-\frac{v^2}{c^2}\right) = \overline{t}_A'\sqrt{1-\frac{v^2}{c^2}}, \text{ q.e.d.}$$

The same result is found if either B or A is supposed moving; the motion is relative hut the phenomenon is not symmetrical.

4. CRITICISM OF OPPONENT VIEWS

Dingle (**g**) thinks that he can prove, by means of the above example, that A's and B's times will be the same at their meeting. His argument is: «Consider three clocks, B_1, C_1, D_1, which at the common reading $t = 0$, are together and such that, if D_1 is regarded as being at rest, B_1 and C_1 are moving in opposite directions, each with uniform speed v with respect to D_1. If these clocks are respectively at the origins of co-ordinate systems of which the x-axes all lie along the line of motion and are positive in the direction towards which C_1 is moving, then these systems are related by the Lorentz transformation. There is a fourth clock A_1, at rest at the point (x, 0, 0) in the D_1 system and at the event $E'_1 = (x, 0, 0 - x/c)$ in that system let it emit a beam of light which, on reaching D_1, is reflected back again to A_1, clearly reaching it at the event $E_3' =$ (x, 0, 0, x/c) in the D_1 system. By the Lorentz transformation, $E_1' =$ (xb, 0, 0, — xb/c) in the B_1 system and (x/b, 0, 0,— x/cb) in the C_1 system, where $b = \sqrt{\dfrac{\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)}}$, while $E_3' = (x/b, 0, 0,$ $x/cb)$ in the B_1 system and $(xb, 0, 0, xb/c)$ in the C_1 system. Hence, by Einstein's criterion, the time of the event E_2', at which the light falls on D_1, will be 0 in the D_1 system, $\frac{1}{2}\left(-\frac{xb}{c}+\frac{x}{cb}\right)$ in the B_1 system and $\frac{1}{2}\left(-\frac{x}{cb}+\frac{xb}{c}\right)$ in the C_1 system. Since D_1 is at rest with respect to A_1, D_1 is synchronized with A_1 if it actually reads 0 at this event. Let this be so. Then, by hypothesis, B_1, C_1 and D_1 will be coincident and will all read 0 at the event E_2'. Hence B_1 will be slow and C_1 fast by the same amount, viz. $x\,(1/b - b)/2c$ at the event E_2'».

Crawford (**b**) pointed out clearly the main error in this argument, which invalidates Dingle's «proof». Dingle's calculations are based on a misunderstanding of Einstein's definition of synchronization. Two distant clocks can be synchronized only if they belong to the same inertial system, i.e. if they are at rest with respect to each other (Einstein **a**, p. 894 f; **b**, p. 16, **c**, p. 18, **d**, p. 26). This is forgotten in the application of Einstein's criterion for the event E_2' of the reflection of the light. In the moving system the time of the event E_2' is the mean between the times of departure and returning of the light to the same point (in the corresponding system) namely xb in B_1 and x/b in C_1. Then, as it is expected, the times of the event E_2' are found to be all zero.

Dingle did not try to refute Crawford's criticism but stated his theorem in another way (Dingle, **k**) :

«Let B_1 and C_1 be two observers - cum - clocks moving with the same velocity v away from and towards, respectively, a distant object A_1 regarded as stationary. Then the events on A_1 which B_1 and C_1 regard as simultaneous with their meeting are respectively before and after, by equal amounts, the event on A_1 which an observer on A_1 regards as simultaneous with that meeting. To prove this, introduce as an intermediary an observer-clock D_1, stationary with respect to A_1, and coincident with B_1 and C_1 at their meeting. Let all three clocks, B_1, C_1, D_1, read 0 at that event so that their co-ordinates are related by the Lorentz trasformation. If x is the distance between A_1

and D_1 in their common rest frame, the event E_0 on A_1 which is simultaneous in the D_1 co-ordinate system with the meeting of B_1, C_1, D_1, is $(x, 0)$ in that system. Hence, by the Lorentz transformation, its time in the B_1 system is $\frac{vx}{c^2\alpha}$ and in the C_1 system, $-\frac{vx}{c^2\alpha}$. Now let A_1 be a clock identical in working with B_1 and C_1 but set at random, so that its reading at the event E_0 is T_0, which may be anything at all. Then clearly the readings of A_1 which B_1 and C_1 respectively, regard as simultaneous with their meeting are $T_0 - \frac{vx}{c^2\alpha}$ and $T + \frac{vx}{c^2\alpha}$. A_1 will obviously regard the meeting of B_1 and C_1 as occuring at T_0, for that is his reading for the event which D_1 times at zero. This proves the proposition. It is easily verified that $\frac{vx}{c^2\alpha} = \frac{1}{2}\left(\frac{x}{cb} - \frac{xb}{c}\right)$ of the former proof».

Dingle's results are correct in this case, but they have no connection with our problem ; they prove no proposition about the clock paradox. In fact we are not at all interested to what B_1 and C_1 r e g a r d in A_1 as simultaneous with their meeting. The proplem is what w e r e the readings of A_1, B_1, when they have met and what w i l l b e the readings of A_1 and C_1 when they will meet. From Dingle's own description it follows that, as B_1, C_1, D_1, show at their meeting the same time 0, B_1 was fast with respect to A_1 (if A_1 is synchronized with D_1) when it met it, by the amount $\frac{x}{v}$ $(1 - \alpha)$, $\left(\text{where } \alpha = \sqrt{1 - \frac{v^2}{c^2}}\right)$ while C_1 will be slow with respect to A_1 when it will meet it, by the same amount. In the general case A_1's readings when it meets B_1 and C_1 are : $T_0 - \frac{x}{v}$ and $T_0 + \frac{x}{v}$, while B_1's reading at the first moment is $-\frac{x}{v}\alpha$ and C_1's reading at the second moment is $+\frac{x}{v}\alpha$. The time interval is $\frac{2x}{v}$ according to A_1, while it is $\frac{2x}{v}\alpha$ according to the B_1-C_1 time (i.e. the time transferred from B_1 to C_1). It is therefore seen clearly that Dingle's example gives the usual asymmetric effect, if rightly described.

A similar case is treated by Dingle geometrically (**g**) and the same conclusions from irrelevant data are drawn. Fig. 1 illustrates the previous description (only the subscripts of ABC have been ommitted).

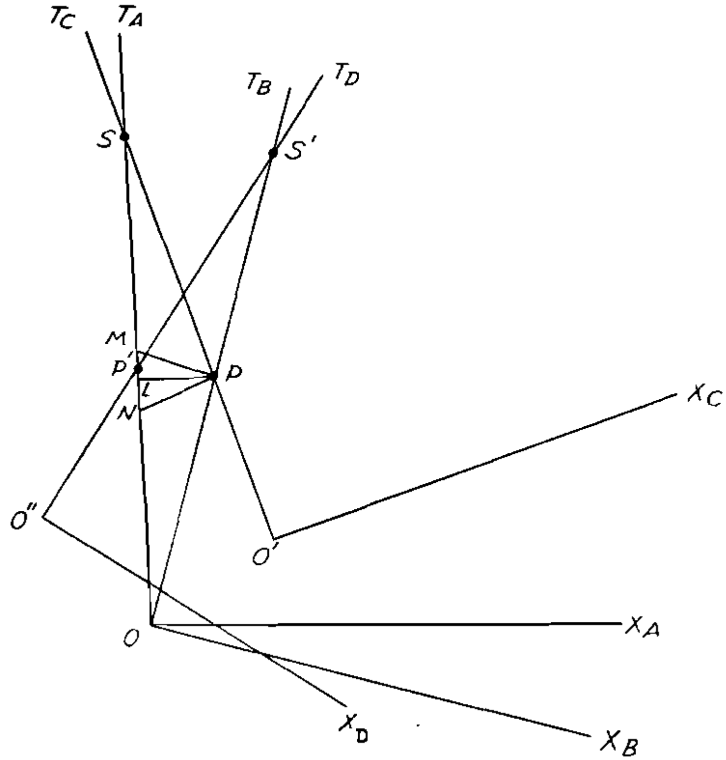


Fig. 1. Geometrical solution of the clock paradox.

The world - lines of A, B, C, are T_A, T_B, T_C respectively ; T_B and T_C form the same angle with T_A. It is seen that the reading of A when B and C meet (simnltaneous with the event P in the system of A) is OL. In the systems of B and C however the readings of A simultaneous with P are OM and ON respectively. It is evidently LM=LN, «i.e., says Dingle, C will regard A as being fast by the same amount that B will regard it as being slow». This however again has nothing to do with the problem of the clock paradox. The problem is not if B of C will regard A as being fast or slow, but if C's reading when it meets A, at the event S, is equal or not with A's reading. It is evident that if A and B have the same readings 0 at O, and if at the event P, B transmits its time to C, then C's time at S is represented by the length (OP) + (PS), which is greater than (OS) ; therefore as the time axis is imaginary C's time is smaller than A's at S. This conclusion is absolute, because the four-dimensional space is absolute.

The symmetric experiment for B involves a fourth clock D which meets A at the event P′ (where OP′=OP) and moves with velocity v with respect to B (i.e. T_D makes the same angle with T_B, as T_B with T_A). If D is synchronized with A at P′, it will show a time represented by (OP′) + (P′S′) at S′, which is smaller than B's time, which is represented by (OS′).

Cullwick has made an errror similar to that of Dingle in his description of the clock paradox (p. 70 - 73 and 283 - 289). He assigns a time t_2 to a distant clock which is moving with respect to our clock, through the relation $t_2 = \frac{t_1+t_3}{2}$ where t_1, t_3 are the times of emission and returning of a light ray which is reflected at this distant clock. This again is not consistent with Einstein's definition of synchronization ; as it is known, only two possibilities to synchronize two clocks exist : either they belong to the same inertial system, or they go through the same point when they are synchronized. Cullwick's t_2 time has no physical meaning, if the distant clock is moving with respect to us; it would give the right time only if it belonged to our own system.* Therefore the so-called «discontinuity» of the coordinate time calculated by A, which according to Dingle (**i**, **q**) and Cullwick takes place when the receding clock B meets the approching clock C, is due only to the fact that these authors try to synchronize two moving distant clocks.

Dingle (**q**) has tried to give another justification also to his claim that no asymmetric ageing will be present when two identical clocks R and M will re - unite after «M's» travel to a certain distance (without using the discontinnity of the time mentioned above). He considers a train M moving from a station A to another station B. The length of the train at rest is equal to (AB) = L. Then

* Further Cullwick bases his arguments on the assumption (p. 73) that if two clocks A and B, belonging to the same system S, are synchronized, then a third clock O′ moving from A to B, shows the same time T′=T as B when meeting B, if it has been synchronized with A when it met it. This is evidently wrong ; in fact, if T is the reading of B (B is synchronized with A) when O′ meets it, O′'s reading is $T' = T\sqrt{1-\frac{v^2}{c^2}}$. This a most elementary application of the Lorentz formulae (see e. g. von Laue **b**, p. 58) and it is strange to see how it has been overlooked.

Dingle adds : «We may then regard the outward journey as ending when the guard's van, G, at the rear of the train arrives at A». This is obviously not the case. A's time, when M arrives at B, is $t = \frac{L}{v}$; then G's coordinate x is given by the Lorentz formula : $x' = \frac{x-vt}{\alpha}$ where $\alpha = \sqrt{1 - \frac{v^2}{c^2}}$, and $x' = - L$, $t = \frac{L}{v}$, therefore $x = L - L\alpha$, i.e. G is beyond the point A. Thus when the jonrney ends, G does not arrive at A. Dingle's description is referring to two symmetric experiments : 1) The front of a train MG (of length L in its own system) moves from A until G reaches A, 2) The front of a train AB (of length L in its own system) moves from M nntil B reaches M (or M reaches B). These two cases are symmetric, i.e. the corresponding time readings are connected by inverse functions in the two cases. But in is not the same experiment described from two points of view, as thinks Dingle.

We have seen that the arguments of Dingle and his followers are not right. How then is the controversy still continuing ? One reason is, I think, that many of Dingle's opponents are also wrong. Dingle's criticism of most of his opponents is in many cases right, because his opponent's argnments are not proving, in general, the asymmetry of the motion of the one clock with respect to the other ; they find, in general, only the usnal contraction of time of a moving observer. Some inadequate expositions have been set out recently by Fremlin (1957), Darwin (1957) and Builder (**a, b, c**). Dingle's criticism to them (**m, n, o, p, q**) is justified because no one gives a convincing argument for the asymmetry between the two clocks *.

5. THE CLOCK PARADOX IN GENERAL RELATIVITY

In this paragraph we follow in general Möller's discussion of the clock paradox, which gives a complete solution of this problem by the General Theory of Relativity.

The clock C_1 is supposed at rest in the origin O of an inertial system S (X, Y, Z, T), while C_2 moves along a straight line to a certain point C of S and then it retnrns to O. At time $T = 0$, C_2

* Dingle (**d, e, f, g**) criticises also a paper by McCrea (**a**); the weak point however is not that suggested by Dingle ; it is McCrea's neglect to prove that the R's time increases from $2\alpha T$ to $2T$ during the acceleration.

begins to be accelerated by a constant force F along the positive X - axis. After the time $\Delta' T$ it reaches a point A with a certain velocity v ; then F is eliminated and C_2 continues its motion with uniform velocity for a time $\Delta'' T$, until it reaches a certain point B. Then an opposite force F brings C_2 at rest to C_1 after a time $\Delta''' T$, which is equal to $\Delta' T$, for symmetry reasons. The return travel is just the opposite of the above. The total time lapse of the travel according to C_1 is $\Delta T_1 = 2\,(2\,\Delta' T + \Delta'' T)$. The corresponding proper time interval of C_2 is : $\Delta\tau_2 = 2\,(2\,\tau_2' + \tau_2'')$. For the first part of the journey OA it is : $F = m_0 g = \frac{d(mu)}{dt} = m_0 \frac{d}{dt}\left(\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\right)$,

where u is the velocity at the time T. Hence $gT = \frac{u}{\sqrt{1-\frac{u^2}{c^2}}}$, as the initial velocity for $T = 0$, is $u = 0$. u increases until it becomes equal to v at time $\Delta' T$; therefore : $g\Delta' T = \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$. The corresponding time $\tau_2' = \int_0^{\Delta' T} \sqrt{1-\frac{u^2}{c^2}}\, dT = \int_0^{\Delta' T} \frac{dT}{\sqrt{1+\left(\frac{gT}{c}\right)^2}} = \frac{c}{g} \sinh^{-1} \frac{g\Delta' T}{c}$;

then it is found $\frac{g\Delta' T}{c} = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \sinh \frac{g\tau_2'}{c}$.

Further $\tau_2'' = \Delta'' T \sqrt{1-\frac{v^2}{c^2}}$. It is easily seen that for $g \longrightarrow \infty$, $\Delta' T \longrightarrow 0$, and $\tau_2' \longrightarrow 0$, i.e. $\Delta\tau_1 \longrightarrow 2\,\Delta'' T$ and $\Delta\tau_2 \longrightarrow 2\,\tau_2'' \longrightarrow$ $\longrightarrow \Delta\tau_1 \sqrt{1-\frac{v^2}{c^2}}$.

If now C_2 is supposed at rest, C_1 is regarded as moving along the negative X axis. While C_1 is accelerated, until it reaches the velocity $-v$, a gravitational field acts on C_1 and C_2, and C_1 is falling freely in it. In order to find the potential of this field, we start from C_1' s inertial system (X, Y, Z, T). With respect to it C_2 moves with velocity $u = \frac{gT}{\sqrt{1+\left(\frac{gT}{c}\right)^2}} = \frac{dX}{dT}$, hence $X = \frac{c^2}{g}\left(\sqrt{1+\left(\frac{gT}{c}\right)^2} - 1\right)$,

bacause for $T = 0$, it is $X = 0$, $u = 0$. Further if t is C_2's proper time, then : $dt = \sqrt{1 - \frac{u^2}{c^2}}\, dT = \frac{dT}{\sqrt{1 + \left(\frac{gT}{c}\right)^2}}$, hence

$t = \frac{c}{g} \sinh^{-1} \frac{gT}{c}$ or $T = \frac{c}{g} \sinh \frac{gt}{c}$. For a point x on the C_2 system (i.e. for a system with origin C_2 and moving together with C_2) we have :

$$X_1 = X + \frac{x}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{c^2}{g}\left(\sqrt{1 + \left(\frac{gT}{c}\right)^2} - 1\right) +$$

$$+ x \sqrt{1 + \left(\frac{gT}{c}\right)^2} = \frac{c^2}{g}\left(\cosh \frac{gt}{c} - 1\right) + x \cos \frac{gt}{c}$$

and $T_1 = T + \frac{u}{c^2} \frac{x}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{c}{g} \sinh \frac{gt}{c} + \frac{gTx}{c^2} = \sinh \frac{gt}{c}\left(\frac{c}{g} + \frac{x}{c}\right)$.

Hence : $ds^2 = dX_1^2 - c^2\, dT_1^2 = \left[\left(\frac{c^2}{g} + x\right) \frac{g}{c} \sinh \frac{gt}{c}\, dt +\right.$

$\left. + \cosh \frac{gt}{c}\, dx\right]^2 - c^2 \left[\left(\frac{c}{g} + \frac{x}{c}\right) \frac{g}{c} \cosh \frac{gt}{c}\, dt + \frac{1}{c} \sinh \frac{gt}{c}\, dx\right]^2 =$

$= dx^2 - c^2\, dt^2 \left(1 + \frac{gx}{c^2}\right)^2$. Therefore the rate of a proper clock at a distance x from the clock C_2 is given by $d\tau_1 = dt\left(1 + \frac{gx}{c}\right)$.

C_1 is falling freely in this field ; for the acceleration period it is $x \simeq O$ and its proper time increases by $\tau_1' = \tau_2' = \frac{v}{g\sqrt{1 - \frac{v^2}{c^2}}}$.

During the constant velocity period it increases by

$\tau_1'' = \tau_2'' \sqrt{1 - \frac{v^2}{c^2}}$. During the decelaration period it is:

$$\bar{x} \simeq -\frac{c^2}{g}\left[\left(\sqrt{1 + \left(\frac{g\tau_2'}{c}\right)^2} - 1\right) + \left(\sqrt{1 + \left(\frac{g\tau_2'''}{c}\right)^2} - 1\right)\right] - v\tau_2''$$

$$\text{and} \quad \tau_1''' = \tau_2'''\left(1 + \frac{g\bar{x}}{c^2}\right).$$

Then : $\Delta\tau_1 = 2\,(\tau_1' + \tau_1'' + \tau_1''')$ and $\Delta\tau_2 = 2\,(\tau_2' + \tau_2'' + \tau_2''')$ with $\tau_2' = \tau_2'''$ for symmetry reasons. When $g \longrightarrow -\infty$, then :

$$\tau_1' = \tau_2' = \tau_2''' \longrightarrow 0,\ \bar{x} \longrightarrow -v\tau_2'',$$

$$\tau_1''' \longrightarrow -\frac{g\tau_2'}{c^2} \cdot v\tau_2'' = \frac{v^2}{c^2} \frac{\tau_2''}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence $\Delta\tau_2 \longrightarrow 2\,\tau_2''$ and $\Delta\tau_1 \longrightarrow 2\,\tau_2''\sqrt{1-\frac{v^2}{c^2}} + 2\,\tau_2''\frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} =$

$= 2\,\frac{\tau_2''}{\sqrt{1-\frac{v^2}{c^2}}}$, i.e. $\Delta\tau_2 \longrightarrow \Delta\tau_1\sqrt{1-\frac{v^2}{c^2}}$ which is the same result as above.

This constitutes a complete solution of the clock paradox. It involves a third body acceleration, since the clock C_1 is supposed at rest with respect to the rest of the world, while C_2 is moving with respect to it. If we want to regard C_1 as moving, a «gravitational» field is needed to be introduced, which acts on the whole universe. All the bodies fall freely in this field except C_2, which is supposed at rest. After the period of the uniform motion, when again an inverted field is introduced, the bodies C_1 and C_1 are separated. The action of this new field upon C_1 and C_2 is not symmetric. Therefore an asymmetry arises, which is reflected in the asymmetric ageing of the two clocks.* Another similar discussion of the clock paradox by the General Theory of Relativity has been made in the case of circular motions around a central mass, by Prof. McCrea (**f**). This problem includes a discussion of the gravitational red - shift at the same time. The results concerning the time dilatation are similar to those found above, i.e. the effect is not symmetric between the two clocks.

6. EXPERIMENTAL VERIFICATION

In 1957 Crawford (**a**) described an experiment which proves the asymmetric ageing of two clocks: It is already known that mesons moving with great velocity have a greater mean life - time than mesons at rest. Many authors (Rossi, Hilberry and Hoag, Rossi and Hall, Rasetti, Rossi and Nereson **a, b**) have found a mean life - time of about 2×10^{-6} sec for μ-mesons at rest (see also Thorndike p. 95 - 108) which, owing to the relativistic time dilatation,

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* Crawford (**b**) is essentially right in his discussion of the problem. Dingle (**h,k**) objects to this the example of the apparent diminution of the size of an observer which is moving away. This however is quite irrelevant, since an approaching observer i n c r e a s e s in size, while the rate of a returning clock is the same as that of a receding one i.e. slower than the rate of a clock at rest.

becomes up to 30×10^{-8} sec for fast mesons i.e. 15 times greater. These experiments do not prove of course the asymmetric ageing, because the observer measures the life-time of the mesons, which lie in another inertial system than his own.

Crawford refers to an experiment made by Ticho at 11.500ft (Ticho, 1947) and at 600ft (private communication to F. S. Crawford) with the same apparatus on decelerated mesons. Fast incident mesons acted as triggers for the counter system, which measured the delayed decays of mesons that were decelerated to rest by an absorber. Ticho could measure the percentage p of the mesons which stopped at the absorber. We may describe his experiment as follows: Suppose that T sec is the half-life of the mesons at rest and that nT seconds after the impact of a pack of mesons on the lower station there remains a percentage of pq mesons (which may be measured from the number of mesons decaying in T seconds after the first nT seconds). Then $q = \frac{1}{2^{n'}}$ where n′ is the increase of the proper time of the mesons during the n seconds of deceleration. If the mesons want mT seconds in our system to move from the higher to the lower station, then the percentage of the remaining mesons is $\frac{p}{2^{m'+n'}}$ of the initial flux, where $m' = m\sqrt{1 - \frac{v^2}{c^2}}$.

The same initial flux of mesons arrives at the high altitude station. In nT seconds these mesons have lived n_1'T proper seconds and remain afterwards at rest. Then after m more seconds there remains a percentage of $\frac{p}{2^{m+n_1'}}$ mesons; this number must be equal to $\frac{p}{2^{m'+n'}}$ if no asymmetric ageing exists. But m is much greater than m′ if v is great enough. Therefore n′ must be greater than n_1'. In Ticho's experiments it should be greater by a factor of about 40. No such effect has been observed; on the contrary it has been found, to the limit of the accuracy used, that $n' = n_1'$. Therefore there exists an asymmetric ageing of the mesons, as their proper time increases by $(m' + n')$ T and $(m + n_1')$ T respectively. The above experiment is equivalent to a return travel of the mesons; because after the mesons have been brought to rest in the same inertial system, they may be brought together with an infinitesimal velocity, so that they should not change their times any more.

In a brief reply to this experiment Dingle (**h**) remarked that a returning meson should lose as much time as it gains in its outward journey. However we may find a sufficiently small velocity, so that if a meson moves with this velocity, the supposed loss (although, as we have seen, it is rather a gain) of time is smaller than any *given* number, without necessitating an infinite duration of the journey, as fears Dingle (**h**, postscript). Dingle has made two further replies to Crawford (**i,k**); he does not question the accuracy of his experiments, but he thinks that «an *experimental* proof of the asymmetry would require that the postulate of relativity must be rejected». We have seen, however, why such a view is not right, and how Dingle's attempts to describe symmetrically the motion of the two moving clocks have failed.

Some authors have proposed experiments in order to verify the asymmetric ageing of the moving mesons (Martinelli and Panofsky, Herman, Cochran) *. They refer to artificially produced mesons which are made to move in circular paths by special accelerators in magnetic fields. Such experiments are now possible, so that it is probable that the asymmetric time dilatation will be soon confirmed. However the experiment described by Crawford is the first that proves that there is really an asymmetric ageing effect in the moving mesons; this result is a confirmation of the theory of Relativity, and a rejection of Dingle's principle of symmetry.

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* Another method is proposed by Singer (**a, b**); he calculates that the difference between the rates of a clock on the earth and on an artificial satellite is $3 : 10^{9}$. This effect should be measured by atomic clocks. However the technique of this experiment seems to be so difficult, as to be not realisable for the present.

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