Sci. Annals, Fac. Phys. & Mathem., Univ. Thessaloniki, **16**, 13 (1976)

REDUCIBILITY t-SIMILARITY t_∞ - SIMILARITY OF DIFFERENCE EQUATIONS

by

J. SCHINAS

(Department of Mathematics, Aristotle University of Thessaloniki)

(Introduced by Prof. G. Georganopoulos)

(Received 22.1.76)

Abstract : *Roughly speaking two linear difference equations are called t - similar if one can be transformed to the other by an invertible linear transformation S(t). If moreover, the transformed equation is autonomous the original one is called reducible. More generally two linear difference equations are called t_∞ - similar, if the series of general term $\| S(t)B(t) - A(t)S(t-1) \|$, where A(t), B(t) are the mappings of the left sides of the corresponding difference equations and S(t) the transformation, is convergent. The following proposition is proved. «The uniform stability of linear difference equations is an invariance property under t_∞ - similarity». From the proposition some remarkable corollaries are obtained. Among them the well known discrete analogue of the theorem of Dini - Hukura - Caligo is included.*

It is well known [2] that a linear periodic difference equation

$$y(t) = A(t)y(t-1); \qquad A(t+p) = A(t),$$

can be transformed, by a linear periodic transformation (Liapunov transformation) into an autonomous difference equation. In the following we are going to study a class of linear difference equations which includes the periodic ones and in particular autonomous equations.

The methods of the proofs and the results obtained are analogous to those for differential and functional-differential equations[3].

Firtsly we introduce some notation. Let $I(\alpha) = \{\alpha, \alpha+1, ...\}$, where α is any fixed integer; $I(t_0) = \{t_0, t_0+1, ...\}$, $t_0 \in I(\alpha)$; E a Banach space with norm $|\cdot|$; L(E) the linear space of linear continuous mappings of E with norm $\|\cdot\|$ induced by $|\cdot|$; $L_h(E)$ the space of (bounded with bounded inverse) linear homeomorphisms of E; I the indentity mapping of E and finally O the zero mapping of E.

2. Let $A: I(\alpha) \to L(E) : t \to A(t)$ be given.

Definition 1. A linear difference equation

$$y(t) = A(t)y(t-1), \qquad t \in I(\alpha+1), \tag{1}$$

is called reducible if there exists a mapping $S: I(\alpha) \to L_h(E)$ such that the substitution

$$y(t) = S(t)z(t) \text{ or } z(t) = S(t)y(t),$$

transforms (1) to the autonomous equation

$$z(t) = Cz(t-1),$$

where C is a constant mapping of E.

Obviously the reducibility of difference equations is an equivalence relation.

Definition 2. Two mappings $A: I(t_0) \to L(E)$, $B: I(t_0) \to L(E)$ are called t—similar if there exists a mapping $S: I(t_0+1) \to L_h(E)$ such that, $\forall t \in I(t_0+1)$

$$B(t) = S^{-1}(t)A(t)S(t-1) \qquad \text{or} \qquad A(t) = S^{-1}(t)B(t)S(t-1).$$

The t—similarity is an equivalence relation and it concides with reducibility, if $B(t) = C$.

Consider the difference equation

$$x(t) = B(t)x(t-1), \qquad t \in I(\alpha+1). \tag{2}$$

The following theorem is the discrete analogue of the continuous case of differential equations and so the proof of it is omitted[3].

Theorem 1. If $A: I(\alpha) \to L(E)$, $B: I(\alpha) \to L(E)$ are t - similar, then (1) and (2) have the same stability properties.

3. *Definition 3.* Two mappings $A: I(\alpha) \to L(E)$, $B: I(\alpha) \to L(E)$ are called t_∞—similar if there exist a mapping $S: I(\alpha+1) \to L_h(E)$ and a mapping $F: I(\alpha) \to L(E)$ satisfying

$$\sum_{s=\alpha}^{\infty} \| F(s) \| < \infty ,$$

such that

$$S(t)B(t) - A(t)S(t-1) = F(t). \qquad (3)$$

Remark 1. The t_∞ — similarity ia an equivalence relation.

Remark 2. If, $\forall t \in I(\alpha)$, $F(t) = 0$, t_∞—similarity becomes t—similarity.

Remark 3. If, $\forall t \in I(\alpha)$, $S(t) = I$, $A: I(\alpha) \to L(E)$, $B: I(\alpha) \to L(E)$, t_∞—similar, then the series

$$\sum_{t=\alpha}^{\infty} \| A(t) - B(t) \|$$

is convergent.

4. The definitions of stability and uniform stability of difference equations, can be found in[1]. From now on we suppose that, $\forall t \in I(\alpha)$, $A(t) \in L_h(E)$. The following lemmas are well known[1].

Lemma 1. The difference equation (1) is uniformly stable if and only if

$$\| Y(t)Y^{-1}(s) \| \leqslant K, \; K \geqslant 1, \; s \in I(\alpha), \; t \in I(s) \qquad (4)$$

where Y(t) is the principal fundamental solution of (1).

Lemma 2. Let $t_0 \in I(\alpha)$, $c \geqslant 0$, $k: I(t_0) \to R^+$ be given. Then any solution y(t) of the scalar inequality

$$y(t) \leqslant c + \sum_{s=t_0}^{\infty} k(s)y(s), \qquad t \in I(t_0),$$

satisfies

$$y(t) \leqslant c \exp \sum_{\upsilon=t_0}^{t-1} k(\upsilon), \qquad t \in I(t_0).$$

We can prove now the following theorem.

Theorem 2. If (1) is uniformly stable and $A, B: I(\alpha) \to L_h(E)$ are t_∞ — similar, then (2) is also uniformly stable.

Proof. Let X(t) be the principal fundameutal solution of (2). Then the mapping $S : I(t_o) \to L_h(E)$ given by

$$S(t) = Y(t)\left[Y^{-1}(t_o)S(t_o)X(t_o) + \right.$$

$$\left. + \sum_{s=t_o+1}^{t} Y^{-1}(s)F(s)X(s\text{-}1)\right] X^{-1}(t)\,, \qquad t \in I(t_o) \qquad (5)$$

is a solution of (3).

In fact, setting it in (3), we obtain

$$S(t)B(t) - A(t)S(t-1) =$$

$$Y(t)\left[Y^{-1}(t_o)S(t_o)X(t_o) + \sum_{s=t_o+1}^{t} Y^{-1}(s)F(s)X(s-1)\right].$$

$$X^{-1}(t)B(t) - A(t)Y(t-1).$$

$$\left[Y^{-1}(t_o)S(t_o)X(t_o) + \sum_{s=t_o+1}^{t-1} Y^{-1}(s)F(s)X(s-1)\right].$$

$$X^{-1}(t-1) =$$

$$Y(t)\left[\sum_{s=t_o+1}^{t} Y^{-1}(s)F(s)X(s-1) - \sum_{s=t_o+1}^{t-1} Y^{-1}(s)F(s)X(s-1)\right].$$

$$X^{-1}(t\text{-}1) = Y(t)Y^{-1}(t)F(t)X(t-1)X^{-1}(t-1) = F(t),$$

which proves that S(t), given by (5), is a solution of (3). From (5) we can also get

$$X(t)X^{-1}(t_o) = S^{-1}(t)Y(t)Y^{-1}(t_o)S(t_o) +$$

$$+ \sum_{s=t_o+1}^{t} S^{-1}(t)Y(t)Y^{-1}(s)F(s)X(s-1)X^{-1}(t_o).$$

Hence, using Lemma 1, we have

$$\| X(t)X^{-1}(t_o) \| \leqslant \| S^{-1}(t) \| \ \| Y(t)Y^{-1}(t_o) \| \ \| S(t_o) \| +$$

$$+ \sum_{s=t_0+1}^{t} \| S^{-1}(t) \| \; \| Y(t)Y^{-1}(s) \| \; \| F(s) \| \; \| X(s-1)X^{-1}(t_0) \|$$

$$\leqslant c_1 + \sum_{s=t_0}^{t-1} c_2 \| F(s+1) \| \; \| X(s)X^{-1}(t_0) \| ,$$

where c_1, $c_2 > 0$ constants. Finally, applying Lemma 2, we find

$$\| X(t)X^{-1}(t_0) \| \leqslant c_1 \exp \sum_{s=t_0}^{t-1} c_2 \| F(s+1) \| \leqslant c_1 \exp \sum_{s=\alpha}^{\infty} c_2 \| F(s) \| .$$

Therefore, by Lemma 1, the result easily follows.

The following corollary is the discrete analogue of Dini-Hukuhara-Caligo theorem for differential equations[3].

Corollary 1. If (1) is uniformly stable and

$$\sum_{s=\alpha}^{\infty} \| B(s) - A(s) \| < \infty ,$$

then (2) is also uniformly stable.

The proof follows easily from Remark 3 and Theorem 2.

Another consequence of Theorem 2 is the following corollary.

Corollary 2. If (1) is uniformly stable, $B : I(\alpha) \to L_h(E)$ and

$$\sum_{s=\alpha+1}^{\infty} \| B(s)B(s) - A(s)B(s-1) \| < \infty , \text{ or } \sum_{s=\alpha+1}^{\infty} \| I - A(s)B^{-1}(s-1) \| < \infty ,$$

then (2) is also uniformly stable.

The proof follows by setting in (3)

$$S(t) = B(t) \qquad \text{or} \qquad S(t) = B^{-1}(t).$$

We know that the principal fundamental solution $X(t)$ of

$$x(t) = Cx(t-1), \qquad , \qquad t \in I(\alpha + 1), \tag{6}$$

where $C \neq 0$ is a constant mapping of E, is given by

$$X(t) = C^{t-\alpha} I \quad , \quad t \in I(\alpha).$$

So

$$X(t)X^{-1}(s) = C^{t-\alpha} C^{-(s-\alpha)} I = C^{t-s} I \quad , \quad s \in I(\alpha) \quad , \quad t \in I(s)$$

and, if (6) is stable,

$$\| X(t)X^{-1}(s) \| = \| C^{t-s} I \| \leqslant K \quad , \quad t \in I(\alpha).$$

Therefore, by Lemma 1, (6) is uniformly stable.

From the above argument and Theorem 1 we have the following corollary.

Corollary 3. If $A : I(\alpha) \rightarrow L_h(E)$, $C \neq 0$ are t_∞ — similar and (6) is stable, then (1) is uniformly stable.

From Definition 1 and Remark 2 we have the following corollary.

Corollary 4. Every stable reducible system is uniformly stable.

Finally we note that stability properties of difference equations are extensively studied in [1, 4] and [5].

REFERENCES

1. De Blasi F. S. and Schinas J., On the stability of difference equations in Banach spaces, Analele stiintifice ale Universitatii, «Al. I. Cusa» — Iasi, Sectia Ia. Matematica, tom XX, (1974), 65 - 80.
2. Hahn W., Stability of Motion, Springer - Verlag Berlin, Heidelberg, New York, 1967.
3. Sansone G. and Conti R., Non-linear differential equations, Pergamon Press, Oxford, 1964.
4. Schinas J., Stability and conditional stability of time-dependent difference equations in Banach spaces, J. Inst. Maths Applcs (1974), 14, 335-346.
5. Sugiyama S., Difference inequalities and their applications to stability problems, Lecture notes in Mathematics, Vol. 243, Springer - Verlag, (1971), 1 - 15.

ΠΕΡΙΛΗΨΙΣ

ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΣ t — ΟΜΟΙΟΤΗΣ t_∞ — ΟΜΟΙΟΤΗΣ ΤΩΝ ΔΙΑΦΟΡΩΝ ΕΞΙΣΩΣΕΩΝ

Ὑπὸ

ΙΩΑΝΝΟΥ ΧΡ. ΣΧΟΙΝΑ

(Ἐπιμελητοῦ τοῦ Μαθηματικοῦ Σπουδαστηρίου τῆς Φ.Μ. Σχολῆς)

Εἰς τὴν παροῦσαν ἐργασίαν μελετῶνται ὡς πρὸς τὴν ὁμοιόμορφον εὐστάθειαν γραμμικαὶ διαφορῶν ἐξισώσεις εἰς χώρους Μπανάχ. Δύο τοιαῦται ἐξισώσεις ὀνομάζονται t — ὅμοιαι ἐὰν ἡ μία μετασχηματίζεται εἰς τὴν ἄλλην δι' ἑνὸς γραμμικοῦ ἐξαρτωμένου ἐκ τοῦ t ἀντιστρεψίμου μετασχηματισμοῦ S(t). Ἐὰν ἐπὶ πλέον ἡ δευτέρα ἐξίσωσις εἶναι αὐτόνομος λέγομεν ὅτι ἡ πρώτη ἀνάγεται εἰς αὐτόνομον. Γενικώτερον δύο γραμμικαὶ διαφορῶν ἐξισώσεις ὀνομάζονται t_∞ — ὅμοιαι ἐὰν ἡ σειρὰ ἡ ἔχουσα γενικὸν ὅρον $\|S(t)B(t) - A(t) S(t-1)\|$, ἔνθα A(t), B(t) αἱ ἀπεικονίσεις ἀντιστοίχως τῶν ἀριστερῶν μελῶν τῶν ἐξισώσεων καὶ S(t) ὁ προαναφερθεὶς μετασχηματισμὸς εἶναι συγκλίνουσα. Ἀποδεικνύεται δὲ τὸ ἑπόμενον θεώρημα: «Ἐὰν δύο γραμμικαὶ διαφορῶν ἐξισώσεις εἶναι t_∞ — ὅμοιαι καὶ ἡ μία ἐκ τούτων εἶναι ὁμοιομόρφως εὐσταθὴς τότε καὶ ἡ ἄλλη εἶναι ἐπίσης ὁμοιομόρφως εὐσταθής». Ἐκ τοῦ θεωρήματος τούτου προκύπτουν ὡρισμένα ἀξιοσημείωτα πορίσματα — μεταξὺ τῶν ὁποίων καὶ τὸ διακεκριμένον ἀνάλογον τοῦ θεωρήματος τῶν Ντίνι — Χουκουχάρα — Γκαλίγκο — διὰ τῶν ὁποίων ἀποδεικνύεται ὅτι ἡ ὁμοιόμορφος εὐστάθεια εἶναι ἀναλλοίωτος εἰς συγκεκριμένας κλάσεις t_∞ — ὁμοίων, t — ὁμοίων ἢ δυναμένων νὰ μετασχηματισθοῦν εἰς αὐτονόμους διαφορῶν ἐξισώσεων.