Sci. Annals, Fac. Phys, & Mathem., Univ. Thessaloniki, **17**, 195, (1977)

ON THE BOUNDED SUBSETS OF A TOPOLOGICAL SPACE

By
N. P. OECONOMIDES
(Department of Mathematics, Aristotle University of Thessaloniki)

(Received 23.6.77)

Abstract: *The notion of boundedness in a topological space, introduced by Sze-Tzen Hu [2], is well-known. A boundedness in a given topological space (X, T) is said to be local if the corresponding universe (see [2], p. 299) is locally bounded. Let* $\{B_i : i \in I\}$ *be the family of all local boundedness in (X, T) and* $B_0(X, T) = \bigcap \{B_i : i \in I\}$ *; it is known [2] that* $B_0(X,T)$ *is a boundedness in (X., T) but it is not necessarily a local boundedness. In this note we give conditions under which* $B_0(X, T)$ *is a local boundedness.*

1. *REMARKS ON* $B_0(X, T)$. In 1968, S. Gagola and M. Gemignani [1] introduced the notion of *«absolutely bounded»* subsets of a topological space (X, T). Later (1973), P. Lambrinos [3] gave a more useful definition of *«bounded»* subsets of (X, T). He proved that the notions of «boundedness» and of «absolute boundedness» are equivalent, and that a subset of (X, T) is «bounded» if and only if it belongs to every local boundedness in (X, T). In other words, $B_0(X, T)$ is the family of all «bounded» (or «absolutely bounded») subsets of (X, T).

According to this, if C(X, T) is the family of all compact subsets of (X, T), $C_0(X, T)=\{W: W \subset C \in C(X, T)\}$ and T^c the family of all closed subsets of (X, T), it is known ([1] and [3]) that:

Proposition 1. (a) $C(X, T) \subset C_0(X, T) \subset B_0(X, T)$

(b) *If* (X, T) *is* T_3 *then* $B_0(X, T) = C_0(X, T)$.

(c) (X, T) *is compact if and only if* $B_0(X, T) = 2^X$.

(d) $T^c \cap B_0(X, T) \subset C(X, T)$.

(e) *If* (X, T) *is* T_2, *then* $T^c \cap B_0(X, T) = C(X, T)$.

In the abstract we said that $B_0(X, T)$ may or may not be a local boundedness in (X, T). We can illustrate this by the following examples.

Example 1. Let N be the set of the natural numbers $T=\{\Phi, N, \{1\}, \{1, 2\},..., \{1, 2,..., n\},...\}$ and $B=\{W: W \subset V \in T\}$. It is easy to see that B_0 $(N, T)=B$ and that B_0 (N, T) is a local boundedness in (N, T).

Example 2. Let Q be tbe set of the rational numbers, U the usual topology for the real numbers and $T=\{V \cap Q : V \in U\}$. It is known that (Q, T) is T_3 and not locally compact. Hencc, according to the theorem 3 of this note, $B_0(Q, T)$ is not a local boundedness in (Q, T).

2. *CONDITIONS.* We know [2] that a subfamily A of a boundedness B in a topological space (X, T) is called *basis* of B if for every $B \in B$ there exists an $A \in A$ such that $B \subset A$ and A is said to be an *open* (*closed, compact*) basis of B if $A \subset T$ $(A \subset T^c, A \subset C(X, T))$.

Theorem 1. $B_0(X, T)$ is a local boundedness in (X, T) if and only if it has an open basis.

Proof. (i) Let the boundedness B_0 (X, T) be local, and $B=\{W: W \subset V \in B_0(X, T) \cap T\}$. It is easy to prove that B is a local boundedness in (X, T), $B_0(X, T) \cap T$ is an open basis of B and $B \subset B_0(X, T)$. Since B is a local boundedness in (X, T), $B_0(X, T) \subset B$. Hence $B_0(X, T)=B$ and consequently B_0 $(X, T) \cap T$ is an open basis of $B_0(X, T)$.

(ii) Conversely, let A be an open basis of $B_0(X, T)$ and $x \in X$. It is obvious that there exists a $B \in B_0$ (X, T) such that $x \in B$ and, since A is an open basis of $B_0(X, T)$, there exists an $A \in A \subset B_0(X, T) \cap T$ such that $B \subset A$. Hence for each $x \in X$ there exists an $A \in B_0(X,T) \cap T$ such that $x \in A$, that is, $B_0(X, T)$ is a local boundedenss in (X, T).

Proposition 2. $B_0(X, T)$ need not have a closed basis, even though it may be a local boundedness in (X, T).

We can illustrate this by the example 1 of the previous paragraph. Indeed, since $T^c = \{N, \Phi, \{2, 3,...\}, \{3, 4,...\},..., \{n+1, n+2,...\},...\}$ it is obvious that B_0 (N, T) has no closed basis, enen though it is a local boundedness in (N, T).

Proposition 3. If (X, T) is locally compact, then B_0 (X, T) has a compact basis.

Proof. Since (X, T) is locally compact it is clear that C_0 (X, T) is a local boundedness in (X, T). Hence $C_0(X, T) \supset B_0(X, T)$ and in view of proposition 1, B_0 $(X, T) = C_0(X, T)$. By definition, $C(X, T)$ is a compact basis of C_0 (X, T) and consequently $B_0(X, T)$ has a compact basis.

Theorem 2. If (X, T) is locally compact, then B_0 (X, T) is a local boundedness in (X, T).

This theorem follows at once from the proof of the previous proposition.

3. *CASE OF T_3 SPACES.* We shall prove that:

Proposition 4. If (X, T) is a T_3 space and B_0 (X, T) is a local boundedness in (X, T), then (X, T) is necessarily locally compact.

Proof. Since B_0 (X, T) is a local boundedness in (X, T), for each $x \in X$ there is a neighbourhood V of x belonging to B_0 (X, T).

On the other hand, since (X, T) is T_3, there is an $A \in T$ such that $x \in A \subset \bar{A} \subset V$. Hence $\bar{A}$ is a neighbourbood of x belonging to B_0 (X,T) and since $\bar{A} \in T^c$, according to proposition 1, $\bar{A}$ is compact. So it bas been proved tbat for each $x \in X$ there is a compact neighbourhood of x, that is (X, T) is locally compact.

Theorem 3. In a T_3 topological space (X, T), B_0 (X, T) is a local boundedness in (X, T) if and only if (X, T) is locally compact.

This tbeorem follows immediately from theorem 2 and proposition 4.

REFERENCES

[1] S. Gagola, M. Gemignani, : «Absolutely bonnded sets», Mathem. Japonicae, Vol. 13, No 2, 1968.

[2] Sze-Tsen Hu, : «Boundedness in a topological space», Journ. de Math. tome XXVIII, Fasc. 4, 1943.

[3] P. Lambrinos, : «A topological notion of boundedness», Manuscripta Math. 10, 289-296, 1973.

ΠΕΡΙΛΗΨΗ

ΠΕΡΙ ΤΩΝ ΦΡΑΓΜΕΝΩΝ ΥΠΟΣΥΝΟΛΩΝ ΤΟΠΟΛΟΓΙΚΟΥ ΧΩΡΟΥ

Ὑπὸ

Ν. ΟΙΚΟΝΟΜΙΔΗ

Εἶναι γνωστὴ ἡ ἔννοια τῆς *δομῆς φραγμένων συνόλων* (boundedness) εἰς τοπολογικὸν χῶρον, ἡ ὁποία ἔχει εἰσαχθεῖ ἀπὸ τὸν Sze-Tzen Hu [2].

Μία δομὴ φραγμένων συνόλων B εἰς τοπολογικὸν χῶρον (X, T) λέγεται *τοπικὴ* ἐὰν δι' ἕκαστον σημεῖον x τοῦ (X, T) ὑπάρχει μία τουλάχιστον περιοχὴ V τοῦ x, ἀνήκουσα εἰς τὴν B.

Ἐὰν $\{B_i : i \in I\}$ εἶναι ἡ οἰκογένεια ὅλων τῶν τοπικῶν δομῶν φραγμένων συνόλων εἰς τὸν (X, T) καὶ

$$B_0(X,T) = \bigcap \{B_i : i \in I\},$$

εἶναι γνωστὸν [2] ὅτι ἡ B_0 (X, T) εἶναι δομὴ φραγμένων συνόλων εἰς τὸν (X, T), δὲν εἶναι ὅμως πάντοτε τοπική.

Εἰς τὴν ἐργασίαν αὐτὴν δίδονται ὡρισμέναι συνθῆκαι, ὑπὸ τὰς ὁποίας ἡ B_0 (X, T) εἶναι τοπικὴ δομὴ φραγμένων συνόλων.