IMPACT OF RHEOLOGY ON MODERN GEODESY

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Abstract

It is seen how the new developments in geodetic sciences concerning the deviations of the Earth's surface from the shape of hydrostatic equilibrium require changes in the approach to the observation and data analysis of some Earth's phenomena and also in the curricula of geodetic studies. The need of studies in theory of elasticity and rheology is here considered.

1. New developments in Geodesy

In the late years '50s two important scientific events gratified Geodesy and opened a new era. I refer to the observation of the variation of the orbital parameters of artifical satellites which allowed an accurate computation of the flattening of the Earth and to the observation of the free oscillations of the Earth which allowed to compute the distribution of density inside the Earth.

All understood the importance of this knowledge in the description of the Earth but, I am afraid, not all understood the far reaching consequences of this knowledge concerning Geodesy itself.

The knowledge of the density distribution, for instance, allowed to check the accuracy of the Radau approximation used in the estimate of the hydrostatic flattening of the Earth or the J_{2H} of the hydrostatic Earth. But the observation of the artificial satellites allowed the computation of the J_2 of the actual Earth.

Since J_{2H} and J_2 are different it was concluded that the Earth is not in hydrostatic equilibrium; the difference of the two bulges at the equator being about 10 meters.

Since it was noted (Caputo 1965) that the oblateness of the Earth equator is of the same order of magnitude it was later concluded that the deviation od the Earth from the hydrostatic equilibrium is not of particular physical significance especially concerning the Earth rotation rate variation in the past, which had been often assumed.

But it was also clear that other ondulations of the geoid were an order of magnitude larger than that of the deviation from hydrostatic equilibrium (Caputo 1965). After eliminating the connection between the hydrostatic flattenig and the rotation rate the question of the origin and support of the others, much more relevant, bulges remained open.

These bulges are obviously due to the topography and to the density anomalies inside the Earth which may be supported by tectonic forces or by elastic forces .

The correlation with tectonic forces, namely convection cells, made by many, was suggestive but never globally convincing.the support of elastic forces implied studies of elasticity and rheology.

That is how time, which left the scene with the rotation rate, entered again in the discussion of the most important and actual geodetic problems.

In fact, with the use rheolgy we may find the right approach to the observation in time and the appropriate study of these geoidal bulges.

To make more clear this point I should mention that we know very little of the Earth rheology, as a matter of fact it is to me most misterious and mostly a subject of speculations; however it has been found that in an anelastic sphere surface ondulations with different wavenumbers have different decay (Caputo 1984a, 1984b, 1987).

The implications of this finding are important for those who observe the ondulations of the geoid in time on the surface of the Earth.

In fact the observations of the variation of height in a fixed point leads to the measure of the combined effect of the different waves forming the bulge without possibility to resolve the contribution of each wave unless the exact shape of the bulge is known and some information on the rheology is known.

Obviously the usual pract_ice to reduce to the same point the observations obtained for different times in different points, after discovering the wavenumber dependence of the rheology, is not admissible and all the discussions of the postglacial uplifts based on this type of data are now open to question.

2. Rheology and Geodesy

In recent years some interest has been given to the problem of inferring the rheology of the mantle from observation of the anelastic rebound of some regions of the Earth which had been subject to the load of ice in glacial periods (for an exaustive bibliography see Körning and Müller (1989)). In these investigation the rheology of the mantle has been assumed according to various models (Körnig and Müller 1989) to reach interesting conclusions and doubts on previous results. In the following we will discuss the problem from a theoretical point of view showing that in an anelastic homogeneous sphere with radius r_o, subject to an axially symmetric surface displacement defined by a zonal harmonic of order n and for a wide class of rheologies including that of Maxwell, the relaxation function depends on n.

The implications on the discussion of some geodetic problems will follow.

3. The relaxation of an elastic sphere

Let the stress τ_{ij} strain ε_{ij} relations be (Caputo 1987)

$$h * \dot{\tau}_{ij} + \mu(\tau_{ij} - \delta_{ij}\tau_{mm}/\beta) = 2\mu\, h * \dot{\varepsilon}_{ij} + \delta_{ij}\lambda\, h * \dot{\varepsilon}_{mm} \qquad (1)$$

where $h(t)$ is a memory mechanism representing the rheology of the sphere, λ and μ are the elastic parameters. Let us also assume that the sphere is subject to a constant surface radial displacement defined by

$$[s_r]_{r=r_0} = D_n P_n(\cos\vartheta) \; ; \; [s_\vartheta]_{r=r_0} = 0 \qquad (2)$$

where $P_n(\cos\vartheta)$ is the Legendre polinomial of order n, ϑ is colatitude. D_n is an amplitude factor and s_r is the radial displacement. The distance from the center of the sphere is indicated with r and s_ϑ is the tangential displacement.

It is seen in the appendix that the Laplace Transform (LT) of the solution assuming $\lambda = \mu$ to simplify the discussion and using capital letters to indicate LT, is

$$S_r = \frac{D_n}{2p}\left(\frac{r}{r_0}\right)^{n-1}\left\{-(n+1)\frac{2(n-1)}{4n+1}\left(\frac{r}{r_0}\right)^2\left[1+\frac{C\,\frac{2n+3}{2(n-1)}}{C+pH}\right] + \frac{2(n+4)n}{4n+1}\left[1+\frac{C\,\frac{2n^2+5n+3}{2n(n+4)}}{C+pH}\right]\right\}P_n(\cos\vartheta)$$

$$S_\theta = \frac{D_n}{2p}\left(\frac{r}{r_0}\right)^{n-1}\left[\frac{2n+1}{4n+1}\left[1+\frac{C\,\frac{(2n+3)(n+1)}{n(2n+1)}}{C+pH}\right]\right]\left(1-\left(\frac{r}{r_0}\right)^2\right)\frac{dP_n}{d\vartheta} \qquad (3)$$

where p is the LT variable and $C = \mu(\lambda + 2\mu/3)/(\lambda + (3n + +1)\mu/n)$.

A wide class of rheologies is represented by $h(t) = \eta\, t^{-z}/\Gamma(1 - z)$, $H = \eta\, p^{z-1}$ (Caputo 1984b) where η = gr cm^{-1} sec^{-2+z} and z is a real number $0<z<1$. This leads to the stress strain relations

$$\eta\, \partial^z \tau_{ij}/\partial t^z + \mu(\tau_{ij} - \delta_{ij}\tau_{mm}/3) = 2\mu\eta\, \partial^z \varepsilon_{ij}/\partial t^z + \delta_{ij}\lambda\eta\, \partial^z \varepsilon_{mm}/\partial t^z$$

In this case the solution of (3), assuming $B = C/\eta$, is

$$S_r = \frac{D_n}{2p}\left(\frac{r}{r_0}\right)^{n-1}\left\{-(n+1)\frac{2(n-1)}{4n+1}\left(\frac{r}{r_0}\right)^2 + \frac{2n(n+4)}{4n+1} + \left[\left(1-\left(\frac{r}{r_0}\right)^2\right)\frac{(2n-3)(n+1)}{4n+1}\,\frac{B}{B+p^z}\right]\right\} P_n(\cos\vartheta)$$

$$S_r = \frac{D_n}{2p}\left(\frac{r}{r_0}\right)^{n-1}\left\{\frac{2n+1}{4n+1}\left(1-\left(\frac{r}{r_0}\right)^2\right) + \left[\frac{2(n+2)^2}{n(4n+1)}\left(1-\left(\frac{r}{r_0}\right)^2\right)\frac{B}{B+p^z}\right]\right\}\frac{dP_n}{d\vartheta} \tag{4}$$

The variable part of the displacement in the sphere due to the surface constant displacement (2) is then (Caputo 1987)

$$\chi(T) = LT^{-1}\left(\frac{B\,p^{-1}}{B+p^z}\right) = \frac{\sin \pi z}{\pi z}\int_0^\infty \frac{1-\exp(-u^{1/z}T)}{u^2+2u\cos\pi z+1}\,du \tag{5}$$

with $T = tB^{1/z}$: the dimensionless time T is measured in units of $B^{1/z}$.

It is thus clear that the part of the displacement in the sphere variable with time depends on n.

In the figures 1 through 5 the function χ (t) is given as function of T. Since $T = tB^{1/z}$ the dependence of χ (t) on z and n is obvious. Since the derivative of the function (5) with respect to n is positive for all values of t the values of the function are larger for larger n or the relaxation of shorter wawenumbers will take shorter time.

The discussion concerning η follows considering that $B^{1/z}$, which in general is smaller than one, decreases with increasing η and therefore for all values of t the relaxation time increases with η if one keeps z constant.

The dependence of the relaxation from z is very relevant for values of z less that 0.1; the table shows that the ratio $\left[B(n = 2)/B(n = \infty)\right]^{1/z}$

z	0.03	0.05	0.1	0.2	0.4	0.6	0.8	0.9
$\left[B(n=2)/B(n=\infty)\right]^{1/z}$	50.7	10.7	3.24	1.80	1.34	1.22	1.16	1.14

For a given $D_n = D$, the same for all n, the assymptotic value of the displacement due to the relaxation is the

$$\frac{D_n}{2}\left(\frac{r}{r_0}\right)^{n-1}\left(1-\left(\frac{r}{r_0}\right)^2\right)\frac{(2n+3)(2n+1)}{4n+1}P_n(\cos\theta)$$

$$\frac{D_n}{2}\left(\frac{r}{r_0}\right)^{n-1}\left(1-\left(\frac{r}{r_0}\right)^2\right)\frac{2(n+2)^2}{n(4n+1)}\frac{dP_n(\cos\theta)}{d\theta} \qquad (6)$$

For any given n, there is alway a value $\bar{r}$ of r

$$\bar{r} = r_0 \exp\left[-(8n^2+4n-7)/(4n^2-3n-1)(2n^2+5n+3)\right] \qquad (7)$$

such that, for any r smaller than $\bar{r}$, the harmonics with order larger than n have an assymptotic value smaller than that of the harmonic n. This implies that in the interior of the sphere the relaxation, in harmonics of higher order, not only occurs faster but it also has smaller assymptotic values.

4.Conclusions

The consequences of the wavenumber dependence of the rheology of the wide class of rheologies considered, if the rheology of the Earth is of this type, are several; among others we note the following:

1) The postglacial rebound observations taken at different points and at different time intervals may not be combined and referred to the same point.

2) The observation of the postglacial rebound at a single point, even for a long time interval, allows only to infer average rheological properties of the different wavenumbers forming the residual deformation.

3) The contributions of the different geoidal ondulations, probably originated at different times, to the departure of the observed J_2 from the hydrostatic one J_{2H}, are decaying with different phases and therefore the difference $J_2 - J_{2H}$ is not significant physically to the discussione of the variation of the Earth's rotation rate.

4) For all z $(0<z<1)$ the relaxation time decreases with increasing n, especially when z is small. This is in agreement with the apparent dominance of the shorter wavelengths in most parts of the topographies of the Earth and of all planets and natural satellites of known topography.

5) The relaxation time decreasing with increasing n favors the dissipation of energy in the longer wavelength and therefore encourages non linear

phenomena and turbulence.

6) The results relative to the assumption $h(t) = \eta\, t^{z}/\Gamma(1-z)$ are immediately extended to the rheologies defined by other functions h(t) such as that of polycrystalline halite (Caputo 1987) for which H(p) is the ratio of two polinomials in p, the numerator of first order and the denominator of second order. Substitution in (3) shows that also in this case the rheology is wavenumber dependent.

Appendix

A class of solutions of the equations of static elasticity, converging in the origin of coordinates is represented by the following uniformly convergent series (Caputo 1961a,1961b)

$$s_r = A_{10} r + \sum_{1}^{\infty}{}_{n} \left(A_{1n} r^{n+1} + A_{3n} r^{n-1} \right) P_n(\cos\vartheta)$$

$$s_\vartheta = \sum_{1}^{\infty}{}_{n} \left(B_{1n} r^{n+1} + B_{3n} r^{n-1} \right) dP_n(\cos\vartheta)/d\vartheta$$

$$B_{1n} = \frac{(n+3)\lambda + (n+5)\mu}{n\lambda + (n-2)\mu} \, \frac{A_{1n}}{n+1}$$

$$B_{3n} = A_{3n}/n$$

(A1)

where $P_n(\cos\vartheta)$ is the Legendre polynomial of order n, s_r and s_ϑ are the radial and tangential components of the displacement respectively, A_{in} and A_{3n} are arbitrary constants to be determined by means of the boundary conditions. Formulae (A1) are a particular case of the more general solution obtained by Caputo (1961b, 1984a, 1987) for a layered elastic sphere expressed again by means of uniformly convergent series.

In the case when the boundary conditions are expressed by (2) we find

$$A_{1n} = \frac{D_n r_0^{-n-1} (n\lambda + (n-2)\mu)(n+1)}{-2n\lambda - (6n+2)\mu}$$

$$A_{3n} = -\frac{D_n r_0^{-n-1} ((n+3)\lambda + (n+5)\mu) n}{-2n\lambda - (6n+2)\mu} \qquad \text{(A2)}$$

If the stres-strain relations are expressed by (1), the LT of the solution of the quasi static equations of elasticity is obtained (Caputo 1987) substituting μ and λ with

$$\lambda \rightarrow [\mu(\lambda + 2\mu/3) + \lambda p H]/(\mu + pH); \quad \mu \rightarrow pH/(\mu + pH) \qquad \text{(A3)}$$

where H = LT(h).

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Figures captions

Fig.1 The function χ(t) with z = 0.1. In the lower scale the value of t corresponds to ((4n + 1)/n)(5η /3μ)) t. In the top scales the values of t correspond to ((4n + 19/n)(5η/3μ)) t with n = 2, n = 10, n = ∞ respectively. It is seen that the function χ (t) reaches faster anv fixed value for larger values of n or that the time of relaxation is decreasing with increasing n. Note that t is in log scale.

Fig.2 As in Fig.1 for z = 0.2.

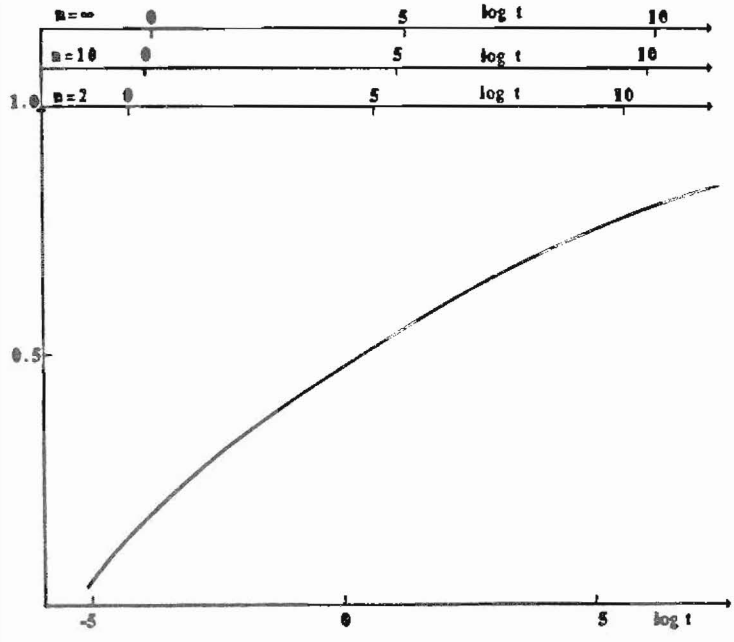
Fig.3 As in Fig.1 for z = 0.4.

Fig.4 As in Fig.1 for z = 0.6. Note that here t is not in log scale.

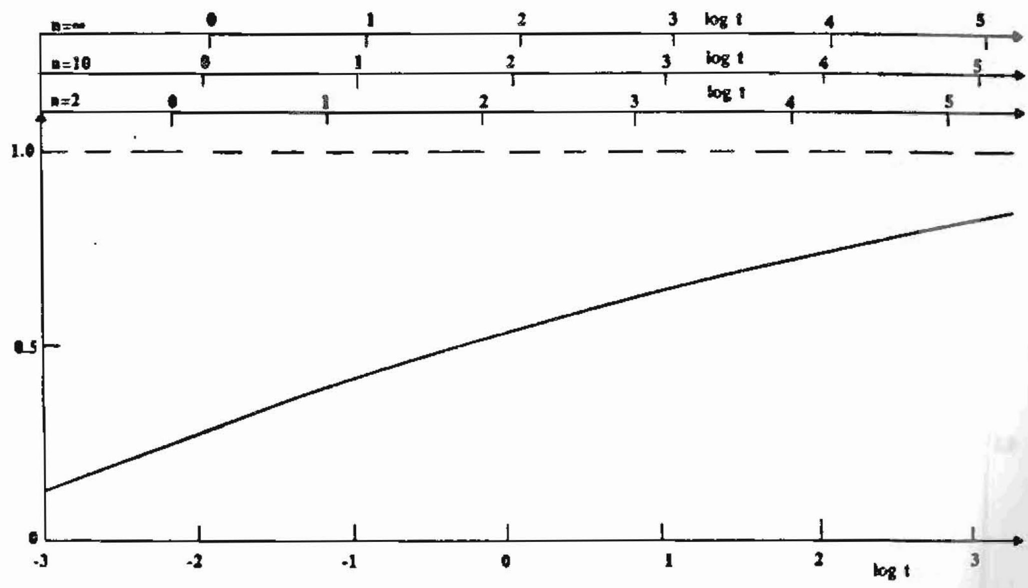
Fig.5 As in Fig.1 for z = 0.8. Note that here t is not in log scale.

Fig.6 Variation of the sea level in the location indicated in the insert map (Fennoscandia), (after Körnig and Müller, 1989).

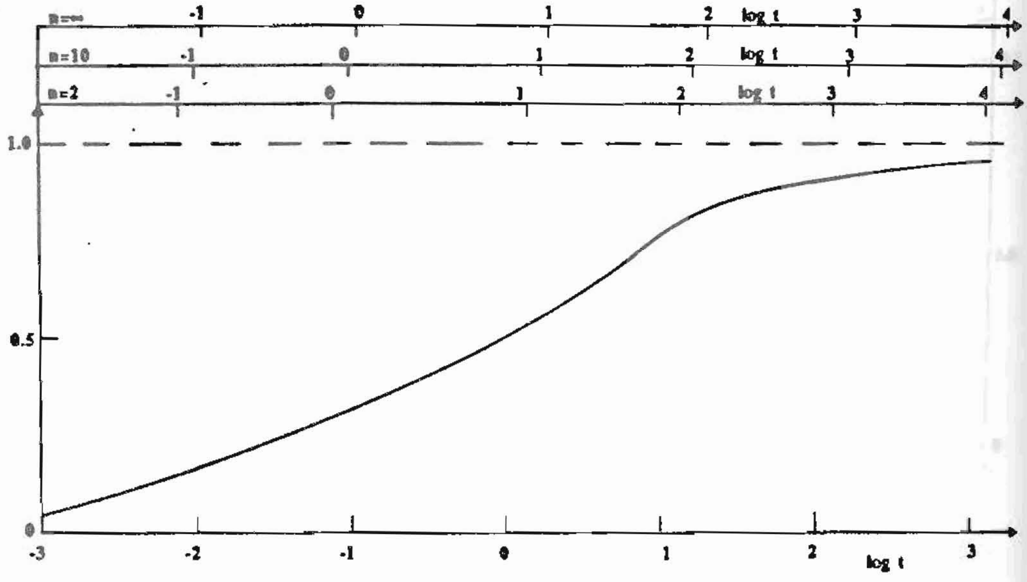
Fig.7 Variation of the sea level in the locations indicated in the insert map (North America), (after Körnig and Müller 1989).

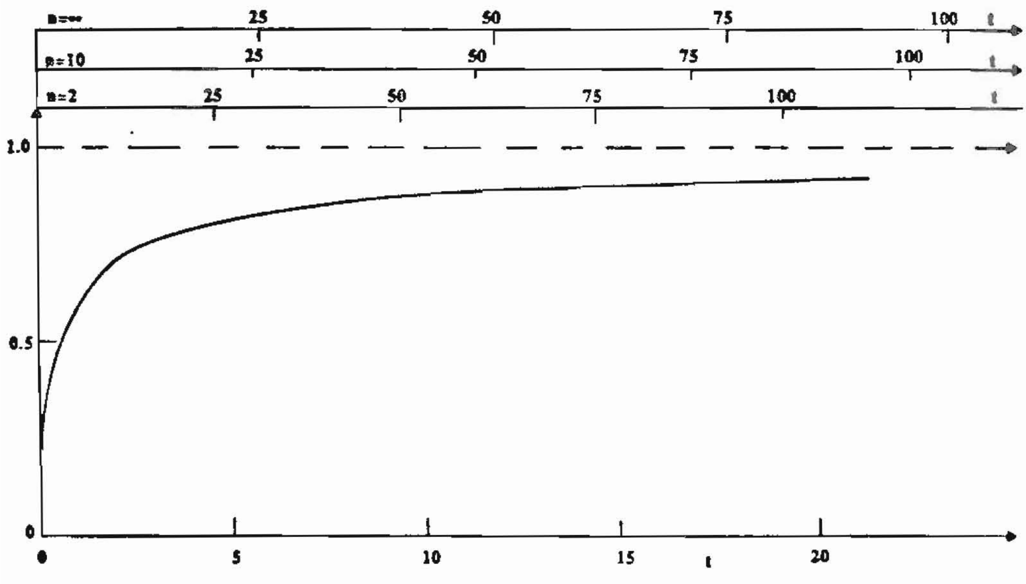


n = ∞
0
5
log t
10
n = 10
0
5
log t
10
1.0
n = 2
0
5
log t
10
0.5
-5
0
5
log t



n=∞
0
1
2
3
log t
4
5
n=10
0
1
2
3
log t
4
5
n=2
0
1
2
3
log t
4
5
1.0
0.5
0
-3
-2
-1
0
1
2
log t
3


n=∞
n=10
n=2
log t
1.0
0.5
0
-3
-2
-1
0
1
2
3
4



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n=2
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50
75
100
t
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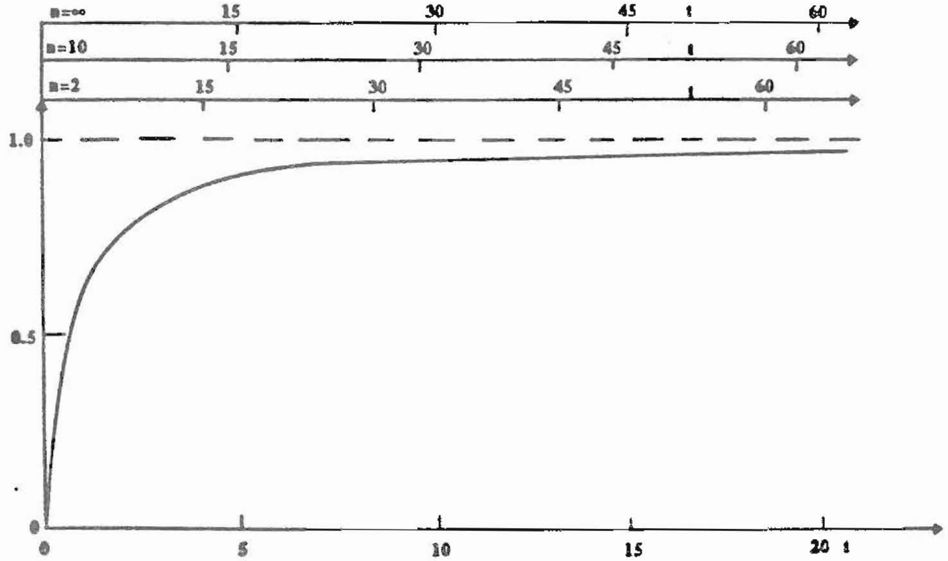
0.5
0
5
10
15
20



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n=2
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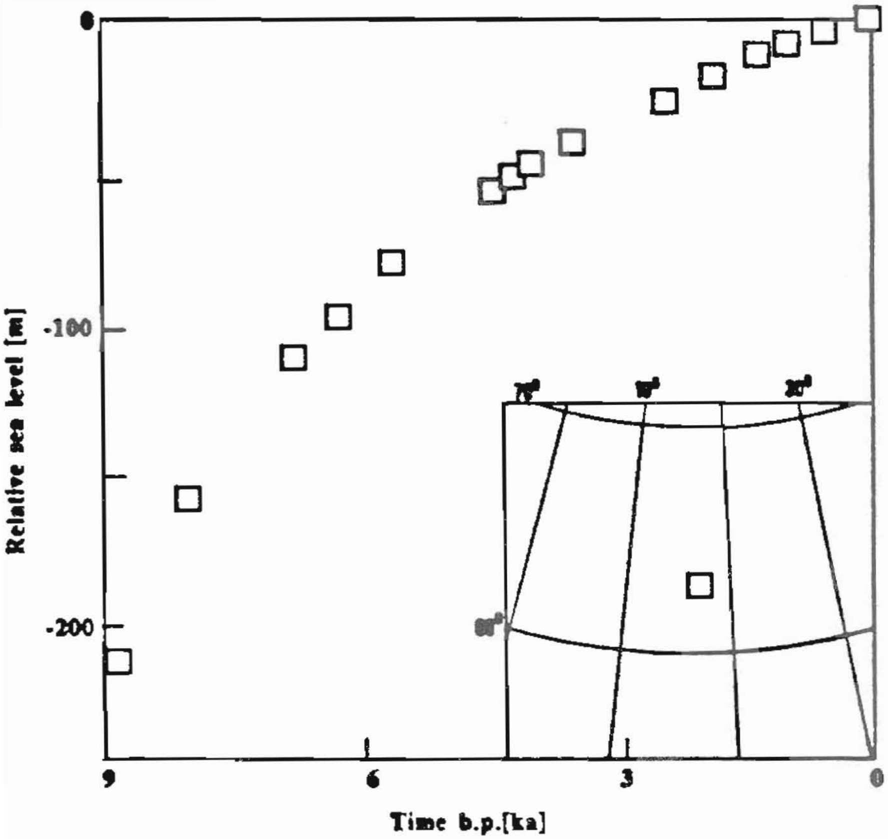
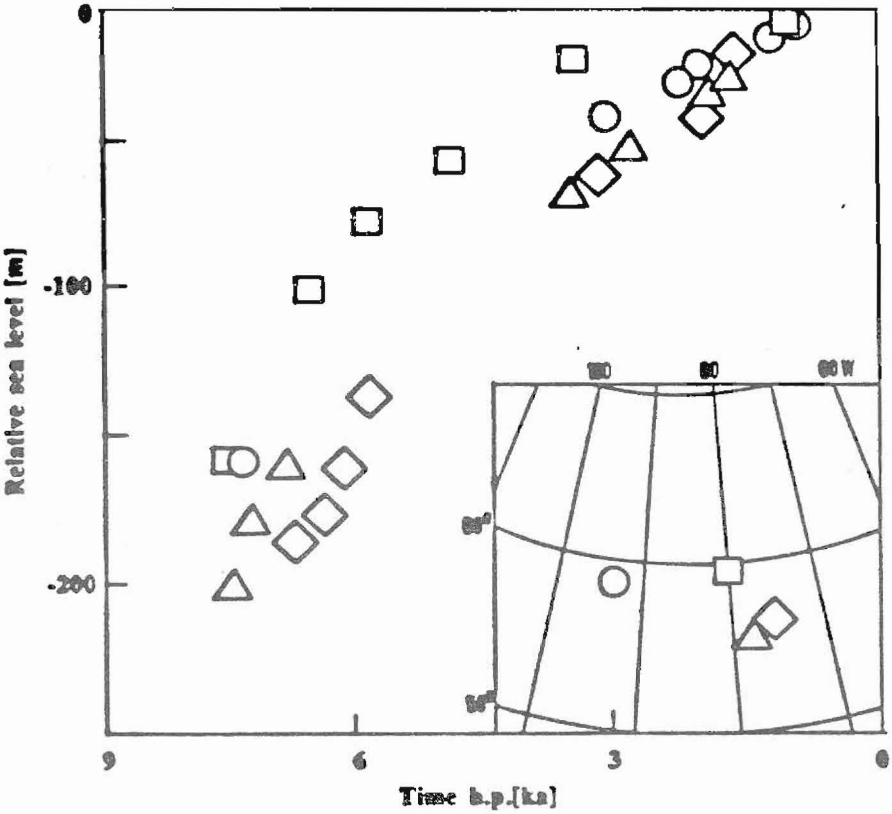
30
45
60
t
1.0
0.5
0
5
10
15
20



Relative sea level [m]
0
-100
-200
9
6
3
0
Time b.p.[ka]


0
-100
-200
Relative sea level [m]
9
6
3
0
Time b.p.[ka]