THE USE OF WEIGHTED BACK-PROJECTION FOR FAST INVERSION OF TWO-DIMENSIONAL RESISTIVITY DATA

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A B S T R A C T

The use of the back-projection (BP) algorithm for the fast inversion of electrical tomographic data is discussed. The algorithm is iterative and is based on the solution of the forward resistivity problem via the finite element method (FEM). The weighting factors used in BP provide an approximation to the sensitivity matrix and are being calculated by a perturbation method.

Two similar BP schemes are proposed: in one the BP is constrained within the region bounded by two equipotential curves while in the second all points are allowed to participate in the BP. Results are presented both for computer generated, synthetic, dipole-dipole pseudosection data as well as for real data. Finally, the theoretical and practical merits and demerits of the method are presented and discussed.

Η ΧΡΗΣΗ ΤΗΣ ΖΥΓΙΣΜΕΝΗΣ ΟΠΙΣΘΟΠΡΟΒΟΛΗΣ ΓΙΑ ΤΗΝ ΤΑΧΕΙΑ ΑΝΤΙΣΤΡΟΦΗ ΔΕΔΟΜΕΝΩΝ ΗΛΕΚΤΡΙΚΗΣ ΤΟΜΟΓΡΑΦΙΑΣ

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Π Ε Ρ Ι Λ Η Ψ Η

Παρουσιάζεται η χρήση της οπισθοπροβολής (ΟΠ) για την ταχεία αντιστροφή δεδομένων ηλεκτρικής τομογραφίας. Ο κυκλικός αλγόριθμος που προτείνεται βασίζεται στην επαναλαμβανόμενη επίλυση του ευθέως ηλεκτρικού προβλήματος με τη βοήθεια της μεθόδου των πεπερασμένων στοιχείων. Οι παράγοντες ζύγισης που χρησιμοποιούνται στην ΟΠ είναι μιά προσέγγιση του πίνακα ευαισθησίας και υπολογίζονται με τη χρήση της μεθόδου διαταραχής. Δύο παρόμοιοι αλγόριθμοι οπισθοπροβολής προτείνονται: στον ένα η ΟΠ περιορίζεται στην περιοχή που βρίσκεται μεταξύ των ισοδυναμικών γραμμών ενώ στον άλλο συμμετέχουν όλα τα σημεία. Παρουσιάζονται αποτελέσματα αντιστροφής τόσο συνθετικών όσο και πραγματικών δεδομένων (ψευδοτομές διπόλου-διπόλου). Τέλος συζητούνται τα θεωρητικά και πρακτικά πλεονεκτήματα και μειονεκτήματα της μεθόδου.

INTRODUCTION

The use of two-dimensional (2-D) resistivity prospection is of considerable current interest due to the development of automatic measuring systems which facilitate the acquisition of a large number of measurements in a limited time (Griffiths et al., 1990). However, the increased quantity of data requires algorithms that can produce subsurface resistivity images in a limited time.

The traditional methods for the interpretation of VP data, such as the construction of a pseudosection provide only a rather qualitative than a quantitative insight into the data. While, the inversion schemes based on iterative least squares algorithms (Smith and Vozoff, 1984) have been shown to give good quality results but are usually quite time consuming.

An alternative strategy towards resistivity inversion is to use approximate schemes, which, despite their intrinsic theoretical weaknesses and limitations, can produce valid sectional resistivity images of the subsurface in a limited time (Barker, 1992).

Related algorithms have been developed for the medical imaging research field, and in particular for a technique called applied potential tomography (APT). APT seeks to reconstruct resistivity images of parts of the human body by the use of electrodes that encircle the area of interest (i.e. human torso). The similarity of the APT reconstruction problem to that of the inversion of multi-probe earth resistivity data is one reason for the study of the applicability of APT algorithms within the geophysical field.

One of the methods widely used in APT is the back-projection (BP) method, which originates from the x-ray tomography field where it is used for solving large linear systems of equations which traditional inversion methods cannot handle. Barber (Barber et al., 1983) proposed a one step algorithm based on BP between the equipotential curves, while Yorkey (Yorkey et al. 1987) used weighted BP in an iterative scheme without constraining the reconstruction between the equipotentials.
The BP algorithms which have been developed for APT are also applicable to linear electrode arrays (Powell et al., 1987), and Noel (Noel and Xu, 1991) developed this for earth resistivity data-sets.

Here, the study of the use of BP for the inversion of multi-probe resistivity data is extended to iterative back-projection schemes. The approach is based on the algorithm suggested by Yorkey and involves the repetitive solution of the forward resistivity problem. For forward modelling the finite element method (FEM) was chosen (Coggon, 1971), since it is the only method that can cope with irregular shapes and boundaries, and therefore can incorporate possible terrain anomalies which could be a significant source of noise within the measurements. The FEM program being used is modified 2.5D version of an original 2D program developed by Dittmer (Dittmer and Szymanski, 1992).

FORWARD MODELLING

General

The forward modelling technique seeks to find a solution of the differential equation that governs the flow of the electrical current in the ground, which is a Poisson's type equation

$$\nabla\cdot(\sigma\nabla V)=\nabla\cdot J \qquad (1)$$

where is the conductivity, V the potential and J the current density. The right hand side of equation (1) here can be replaced by a Dirac delta function and a point current

The basic concept of the FEM is to subdivide the area into subregions (elements) each of which is considered to have a homogeneous conductivity distribution. The unknown potential V_e for each subregion is approximated by a simple function linked to specific points called nodes. The approximate solution for every element e has a power series form:

$$\tilde{V}_e=\sum_{I=1}^{n} a_i\Phi_i \qquad (2)$$

where n is the number of nodes for that element, a is the unknown nodal potential, and φ . are the trial functions which determine the form of the approximation. Both the trial functions and the number of the nodes depend on the type of the element used. Suppose that G(V)=F is the general form of the field equation: by substituting the potential V with its approximation then the residual R caused by the approximation is $R_e = G(V_{ap.})-F$. An optimization criterion should now be defined so as to obtain a set of parameters a_i for which the residual R_e becomes minimal. The most popular and general criterion in FEM analysis is the Galerkin weighted residual method, which states that the weighted average of R_e for each parameter a_i should be zero. The weighting functions are the trial functions associated with each a_i.

$$\int_e R_e\Phi_i=0 \qquad i=1,2,..n \qquad (3)$$

For our problem we used linear triangular elements (three nodes, at the vertices) since they offer a good compromise between simplicity and flexibility. The potential is approximated for the linear triangular element by

$$\Phi_j=\frac{A_j+B_{ix}+C_{jz}}{2D} \qquad (4)$$

For these elements the trial functions are easily derived as

$$\tilde{V}_e=\sum_{j=1}^{3} a_j\phi_j \qquad (5)$$

where D is the area of the element, $A_j=x_kz_l-x_lz_k$, $B_j=z_k-z_l$, $C_j=x_l-x_k$, and x, z are the nodal coordinates. The subscripts j, k, l take initially the values 1, 2, 3, and they change their order cyclically.

2.5 Modelling

In the 2.5D problem the conductivity is allowed to vary in only two dimensions (x,z) while the current distribution is considered to be three-dimensional. Hence, it is assumed that the modelled features extend an infinite distance in the y (strike) direction. Under these assumptions, the expression of the field equation (1) will be

$$\nabla\cdot(-\sigma(x,z)\nabla V(x,y,z))-\sigma(x,z)\frac{\partial^2 V(x,y,z)}{\partial^2 y}=I\delta(x)\delta(y)\delta(z) \qquad (6)$$

(The gradient operator is now only two-dimensional). The potential field varies in three coordinates and hence this dependence should be included in the FEM formulation. The classical way is to Fourier transform the potential variation in the y direction into the wave-number domain. In this way the FEM can find the solution of the transformed potential at a set discrete wave-number points and the potential can then be recovered by applying the inverse transformation. Because of the constant resistivity in the strike direction the potential V(x,y,z) is an even function of y and so we can apply the cosine Fourier transform. The transformed potential is given by

$$V_k(x,k,z)=\int_0^{\infty} V(x,y,z)\cos(ky)\,dy \qquad (7)$$

By replacing V(x,y,z) by $V_k(x,k,z)$ in equation (6) we get

$$\nabla\cdot(-\sigma\nabla V_k+\sigma k^2 V_k=I\delta(x)\delta(z)=f \qquad (8)$$

Hence the initial field equation is now transformed into a Helmholtz's type equation in the wave-number domain.By substituting the approximate solution into equation (8) and by applying the Galerkin method we obtain for the linear triangular elements

$$-\sigma\iint_e\left[\frac{\partial}{\partial x}\left(\frac{\partial\tilde{V}_k}{\partial x}\right)\phi_{ki}+\iint_e\left[\frac{\partial}{\partial z}\left(\frac{\partial\tilde{V}_k}{\partial z}\right)\phi_{ki}\right]+\sigma\iint_e k^2\tilde{V}_k\phi_{ki}\,dxdz=\iint_e f\phi_{ki}\,dxdz$$

(9)

where i=1,2,3, ϕ is the trial solution for the transformed potential and an isotropic medium is assumed. The final step is to integrate by parts and to substitute the trial solution. The numerical evaluation of the integral of equation (9) is straightforward since the trial function terms depend only on the nodal coordinates.

System Equations and Boundary Conditions

Since the left hand side of equation (9)(stiffness terms) can be evaluated from the nodal coordinates the element equations can be assembled into a single set of linear equations, since elements will share common nodes. The resulting global system will have the general form $K\,A_k = F$ where the matrix K contains the stiffness terms and is square, sparse,symmetrical and banded, the vector A_k contains the transformed nodal potential, and the matrix F contains the current source and boundary terms.

After creation of this global system two boundary conditions (BC) are applied: 1) Neumann BC: at the air-earth interface there is no current flow perpendicular to the boundary, 2) Dirichlet BC: the value of the potential at the side and bottom boundaries is zero since these boundaries are designed to be far away from the current source.

The final step is to solve the system of equations using Gaussian elimination to obtain the unknown transformed nodal potential. In order to obtain the potential V(x,y,z,) the inverse Fourier cosine transform should then be applied which for the case of linear arrays (the profile is at y=0) reduces to

$$V(x,y,z) = \frac{2}{\pi}\int_0^{\infty} V(x,k,z)\,dk \qquad (10)$$

Hence, if the transformed potential $V_k(x,k,z)$ is calculated for several k, the potential V(x,y,z) can be obtained by numerical integration of equation (10). The area for $k<=1$ was calculated by Gaussian quadrature using four critical value points and for $k>1$ a least-squares exponential fit was obtained using three points, and the integration then conducted analytically.

In conclusion, the FEM, given a resistivity distribution and a current source will give the potential values at discrete points of the mesh. Consequently, point-to-point potential variations (which are of interest in this case) or apparent resistivity values can be easily recovered.

THE RECONSTRUCTION ALGORITHM

General

The proposed iterative technique seeks to obtain an estimate of the subsurface resistivity distribution for which the predicted measurement values (obtained by the solution of the forward problem) are as close as possible to the measured data. In order to achieve this, an initial resistivity distribution is assumed (usually uniform) and by using the forward modelling technique, the measurements that correspond to this distribution are obtained.

These modelled measurements are compared with the original data and the weighted differences are back-projected in order to obtain a correction to the resistivity estimate. This correction is added to the current resistivity distribution and the procedure is repeated until the difference between the measured and the modelled data is minimal.

Let the M measured data values obtained by using a measuring technique (dipole-dipole in this work) be represented by a vector **D** where $\mathbf{D}^T=\{D_1,D_2,...D_M\}$, and let d be the vector which represents the modelled data, $\mathbf{d}^T=\{d_1,d_2,...d_M\}$. The N subsurface blocks that are allowed to vary their resistivity independently are represented by a vector x with $x^T=\{x_1,x_2,...x_M\}$. The iterative procedure that can be defined is:

1. $\mathbf{x}^k$ is the resistivity estimate at the k iteration;
2. Calculate the modeled data set d^k that corresponds to this resistivity distribution by using the FEM;
3. Find the resistivity correction vector $D\mathbf{x}^k$: the resistivity correction at every block j will be

$$Dx_j^k = \frac{\sum_{i=1}^{M} \frac{(D_i - d_i)}{d_i} P_{ij}}{\sum_{i=1}^{M} |P_{ij}|} x_j^k \qquad (11)$$

where P_{ij} is the weighting factor that corresponds to block j and measurement i;
4. Set $\mathbf{x}^{k+1}= \mathbf{x}^k+D\mathbf{x}^k$
5. Repeat until $D\mathbf{x}^k$ is ''small enough''.

This procedure has similarities to the simultaneous iterative reconstruction technique (SIRT) proposed by Gilbert (Gilbert, 1972) in the sense that the resistivity corrections are taking place after the entire data-set has been back-projected.

Equation (11) describes the back-projection procedure. The final correction for each block is the sum of the weighted differences between the observed and the modelled data. The fractional difference between the observed and the measured data imply a linear relation between the resistivity distribution and the boundary measurements.

Calculation of the Weighting Factors

The weighting factors play a significant role in the BP procedure. Their importance derives from the fact that each parameter contributes to each individual measurement to a different extent. Hence the weighting factor must reflect this varying sensitivity of the parameters towards the measurements. The use of the Jacobian (sensitivity) matrix is the most appropriate choice for this problem. The elements of the MxN Jacobian matrix can be calculated either analytically or numerically.

In this work a perturbation method was used in order to keep the reconstruction time limited. The perturbation technique gives a linear approximation to the Jacobian matrix in a finite difference sense.An initial homogeneous distribution is assumed and the i measurement which corresponds to this homogeneous distribution is calculated. Consequently the resistivity of one block, say j, is perturbed and the new i measurement is obtained. Finally, the ij element of the sensitivity matrix is approximated by the fractional difference between these two measurements. The procedure is repeated for every block and every measurement and in this way the approximate Jacobian matrix is precalculated and stored in a library file so as to be used during the reconstruction.

Constraining Back-Projection

Within the BP method it is quite usual to apply some constraints in order to use only the most significant parts of the Jacobian matrix. However, the ill-conditioned nature of the resistivity method means that discarding the smaller sensitivity coefficients may lead to the loss of vital information hence, a proper threshold should be defined.

One widely used constraint approach is to define a region of the subsurface bounded by the two equipotential curves which terminate at the probes of the measuring probe pair. Only those sensitivity matrix elements which correspond to area within the region are allowed to participate in the reconstruction - all others are set to zero. This assumption is broadly reasonable from a theoretical viewpoint and has been confirmed experimentally.

However, since the other sensitivities are not negligible, it is better to increase the number of the sensitivity matrix entries that are allowed to participate in the reconstruction. Since the Jacobian matrix can be quite large an arbitrary threshold is usually set, in order to restrict the memory and time requirements.

PROGRAM IMPLEMENTATION AND RECONSTRUCTION EXAMPLES

A computer code was written in order to enable the automatic reconstruction of dipole-dipole pseudosection resistivity data, but nevertheless other arrays (e.g. expanded Wenner) can be easily incorporated as well. The sensitivity matrix is precalculated and stored in a look-up file for later use. A relatively small mesh was designed (545 nodes, 1100 elements) and the parameters were placed at the centre of the mesh where the nodal density is high.

The program was tested using modelled data (fig.1 and fig.2). The maximum dipole separation was eight dipole lengths (n=8) and the entire measuring pattern included 13 current dipole positions. The reconstruction of the synthetic data indicated that the iterative scheme can give results which are superior to those of the non-iterative one. Furthermore, it can be seen that the unconstrained scheme performed slightly better than that which constrains the reconstruction between two equipotential lines. However, its potentials are limited due to the use of the approximated Jacobian matrix.

The method was also tested with real data. The first field trial is from the area of Souroti (N. Greece). Dipole-dipole pseudosection data were collected from the region of the fault that is believed to supply the mineral water spring of Souroti. The fault was identified by a previous survey (Vargemezis and Nagoulis, 1992). The measuring scheme used was similar to that used for the modeled data and the length of the dipoles used was 6 metres. The reconstruction results (fig.3) are consistent with the existing VLF data and clearly indicate the fault's position.

The second case study is from the archaeological site of Europos (N.Greece). Based on a resistivity map from the area of the Roman cemetery of the ancient town (fig.4a), dipole-dipole vertical profiling data were collected with dipole spacing of 1 metre. The measured sections were taken over the centre of high resistivity features which are revealed on the map. These features have been interpreted as tombs (Tsokas et al., 1991) and this interpretation was later verified by the excavation results. In fig.4b a reconstruction from a data-set measured over a tomb is shown: the side walls and the floor of the tomb are successfully reconstructed, and the resultant image is in accordance with the excavation results.

DISCUSSION

Because of the assumption of linearity the convergence of the algorithm is not guaranteed, nor can the speed of convergence be predicted. However, in most cases 5-10 iterations are adequate in order to achieve a small RMS error. Moreover, the convergence, if there is convergence at all, can be speeded up by the use of an over-relaxation factor within the reconstruction algorithm. The calculation of the sensitivity matrix at every iteration will almost certainly improve the convergence properties of the scheme.

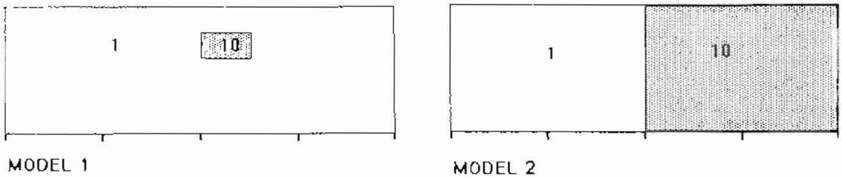

Fig.1. The model structures used to produce VP synthetic data: (model 1) a resistive block which has a resistivity of 10 ohm m. in a medium of 1 ohm m., (model 2) a vertical discontinuity, the medium on the left has a resistivity of 1 ohm m. while the medium on the right has a resistivity of 10 ohm m.)

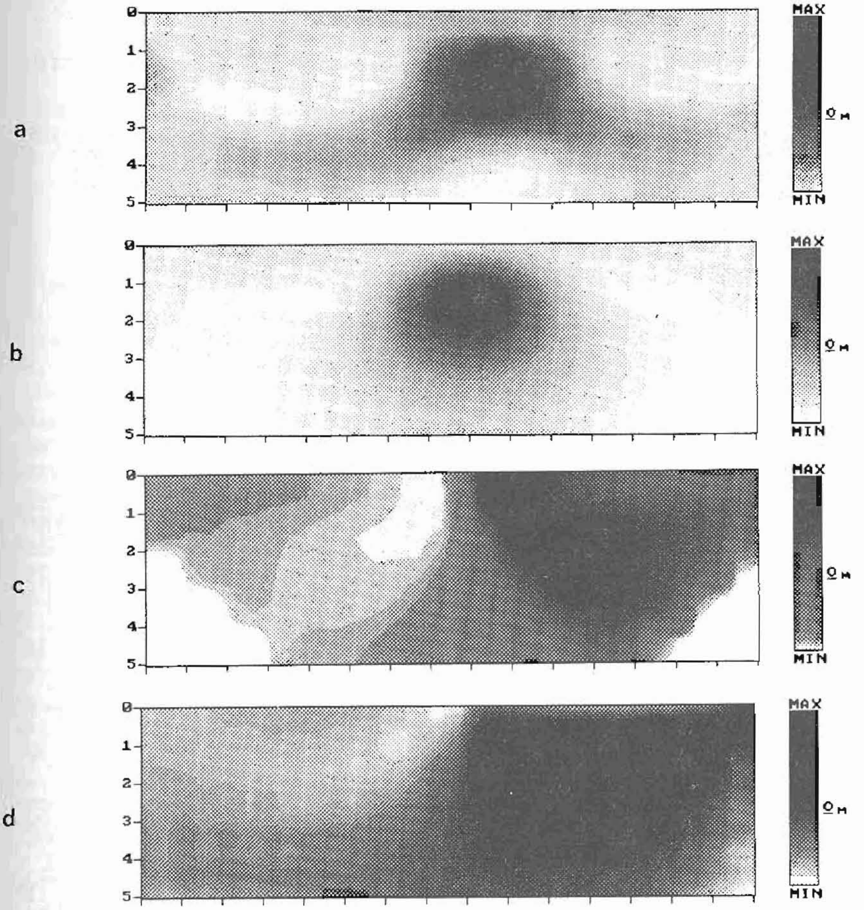

Fig.2. (a) Reconstruction of model 1 using iterative BP constrained between the equipotential curves (10 iterations). (b) Reconstruction of model 1 using iterative unconstrained BP (10 iterations). (c) Reconstruction of model 2 using iterative BP constrained between the equipotential curves (10 iterations). (d) Reconstruction of model 2 using iterative unconstrained BP (10 iterations).

The method gave good results when the intention was only the reconstruction of simple resistivity distributions but, as is quite common in approximate inversion schemes, in more complex situations it did not perform as well. Moreover, sometimes artifacts appeared in the final reconstruction results. These problems derive from the fact that the application of the BP method in non-linear problems such as this is not strictly correct. An additional limitation of the method is that it requires a high density of measurements. Nevertheless, this requirement can be met easily by the use of automatic measuring

The main advantage of the method is that it enables a relatively fast reconstruction of the data giving useful resistivity sectional images. Hence it is believed that it can serve as a practical field tool for the interpretation of data involving shallow depth geological, civil engineering and archaeological features.

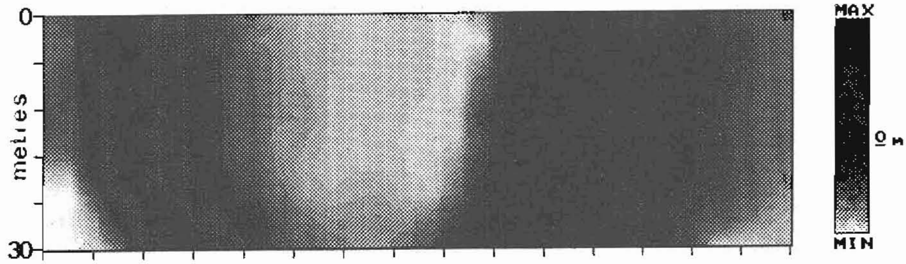

Fig.3. Reconstruction of the data obtained from the area of the fault of the mineral spring of Souroti. The iterative equipotential method was used (5 iterations). Vertical and horizontal scales are identical.

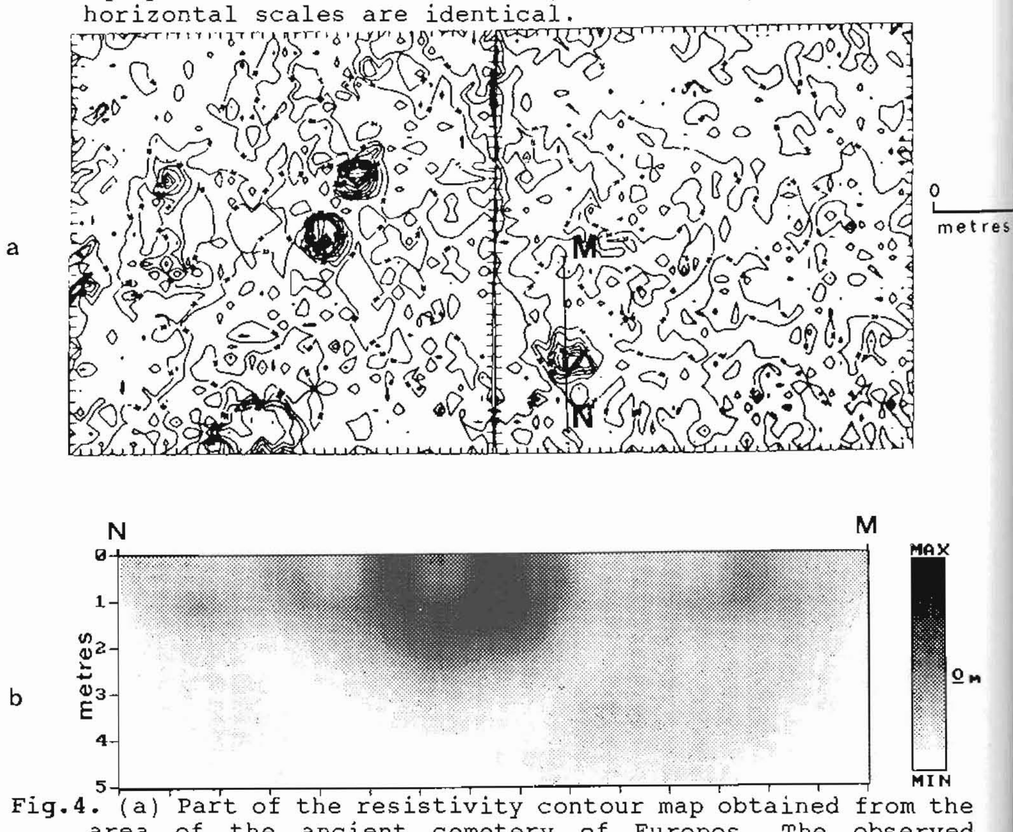

Fig.4. (a) Part of the resistivity contour map obtained from the area of the ancient cemetery of Europos. The observed resistivity anomalies have been interpreted as tombs (Tsokas et al., 1991). (b) Reconstruction of the data, obtained at the section NM (shown in fig. 4a), using the iterative equipotential method (5 iterations).Vertical and horizontal scales are identical.systems when shallow features are surveyed.

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