THREE-DIMENSIONAL DIFFRACTION TOMOGRAPHY FUNCTIONALS

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A B S T R A C T

The basis of interpretation problem is the model of the conection between measurements data being usually digital records and the medium parameter fields. Using the linear model of signal propagation, we construct the linear functionals having the property of the influence function of variation of parameter fields on the digital sample. This influence function is the result of interaction of the incoming (φ_{in}) and inversed outgoing (φ_{out}) sounding signal fields such that the φ_{in} and φ_{out} are determined by the model of the signal propogation in reference medium.

The optimal decision criterion has been investigated and the physical interpretation of tomography functionals has been performed. The information statistical characteristics of the retrieval of the functional values from the field of the medium parameters have been analyzed.

3-D ΣΕΙΣΜΙΚΗ ΤΟΜΟΓΡΑΦΙΑ ΠΕΡΙΘΛΑΣΗΣ

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Π Ε Ρ Ι Λ Η Ψ Η

Η βάση ενός προβλήματος ερμηνείας είναι το μοντέλο το οποίο συνδέει τα πειραματικά δεδομένα, που είναι συνήθως ψηφιοποιημένες καταγραφές και τις επιθυμητές παραμέτρους του μέσου. Χρησιμοποιώντας το γραμμικό μοντέλο διάδοσης του σήματος , κατασκευάσαμε γραμμικά συναρτησιοειδή τα οποία έχουν την ιδιότητα της συνάρτησης επίδρασης της μεταβολής των παραμέτρων στο ψηφιοποιημένο δείγμα. Αυτή η συνάρτηση επίδρασης είναι αποτέλεσμα της αλληλεπίδρασης του εισερχόμενου (φ_{in}) και αντίστροφου εξερχόμενου (φ_{out}) ηχητικού σήματος τέτοια ώστε τα φ_{in} και φ_{out} να καθορίζονται από το μοντέλο διάδοσης του σήματος στο μέσο αναφοράς. Διερευνήθηκε το ιδανικό κριτήριο απόφασης και δόθηκε φυσική ερμηνεία των συναρτησιοειδών της τομογραφίας. Τα στατιστικά χαρακτηριστικά πληροφορίας της ανάκτησης των συναρτησιοειδών απο το πεδίο των παραμέτρων του μέσου μελετήθηκαν και αναλύθηκαν. Μιά κριτική ανάλυση της μεθόδου των Backus-Gilbert (GB) παρουσιάζεται. Για αντιπροσωπευτικό αριθμητικό παράδειγμα χρησιμοποιήσαμε ένα τροποποιημένο αλγόριθμο της τομογραφίας περίθλασης. Αυτός ο αλγόριθμος μπορεί να θεωρηθεί σαν μιά γενικευμένη μέθοδος μεγίστης κλίσης. Ο σκοπός του παραδείγματος ήταν να διευρευνήσει τη διακριτική ικανότητα του

συστήματος παρατήρησης που αποτελούνταν από ένα κανονικό δίκτυο γεωφώνων τριών συνιστωσών.

INTRODUCTION

The application of computerized tomography to the solution of geophysical problems has been based to a great extent on the models of ray propagation of the sounding signals; here, for the retrieval of the fields of medium parameters, various modifications of the Radon inversion have been used (Ivansson and Gustavsson, 1986). Generalized Radon transforms have been constructed for the solution of an inverse dynamic scalar problem (Beilkin, 1985). In recent years, the interpretation of inverse dynamic linearised problems as diffraction tomography problems has become widely practised (Devanie, 1985; Tarantola, 1987; Ryzhikov and Troyan, 1986, 1987).

An inmportant feature of diffraction tomography problems is their incorrectness (instability of solution). Therefore in the solution of inverse problems it is necessary to use regularization methods (Franklin, 1970; Tikhonov, 1963) which are determined by the type of a priory information. In connection with the presence of noise in the measurement data, preference is given to the statistical regularization model.

CONSTRUCTION OF LINEAR FUNCTION OF MEASUREMENTS

When problems of geophysical tomography are set, initial data u (digital samples) for spatially located receivers are used. As a rule, in geophysics the fields of sought-for parameters are elements, $\Theta(\chi)$, of functional spaces $\Theta(\chi)$ $C(R^3)$; for instance, the fields of magnetic, gravitational anomalies, the fields of elastic parameters of Lame $\lambda(\chi)$, $\mu(\chi)$, of density q(x), etc. The measurement space is a N-dimensional Euclidean space (R^N), where N is the number of samples. The tomography experiment is determined by the mapping of the functional space into the measurement space

$$\Theta(R^3)-R^N$$

The experimental data are noise-including values of the n-functionals of the sought-for fields $\Theta(x)$

$$u_n=P_n+\epsilon_n \qquad (1)$$

The tomography experiment consists of the registration of the sounding signals which have passed through the examined medium, and the physical nature of the signals can be different. In many cases of practical interest, geophysical interpretation may be reduced to the consideration of the propagation laws in the linear approximation. Thus, if the sounding impulse field is φ and the source fields is f, we may assume that the process of propagation is described by the linear operator L_Θ:

$$L_{\theta\phi}=f \tag{2}$$

$$L_\theta : L_\theta(\alpha\phi+\beta\psi)=\alpha L_\theta S+\beta L_\theta \psi$$

The operator L_θ determines the medium properties. The problem of interpretation of the tomography experiment from the mathematical point of view is reduced to the retrieval of the operator L_0 from the measurement data. It should be noted that in the physical experiment we register not the field φ proper, but its transformation by the apparatus function of the registering channel H which includes the temporal and amplitude discritization. The most commonly used model of the transformation of this sounding signal by the registering channel is the integral linear operator of convolution. With the known source function f, the formal experimental data model can be written as

$$u_n=H_n L_\theta^{-1} f+e_n \tag{3}$$

$$H: H_n\phi=\int\int\int d\Omega \;\; dt \; dx \; h_n(e_n,e;t_n-t)\delta(x_n-x)\phi(e,x,t)$$

$$e: |e|=1$$

where all the sought-for properties of the three-dimensional medium are included in the operator L_θ^{-1}. It is impossible to find an accurate solution of the problem of retrieval of Θ from the measurement data available. Therefore, the approximate methods are naturally used for the description of the propagation of the sounding signal. Assuming that for a certain value of Θ_o we can build the decision

$$\phi_o=L_\theta^{-1} f \tag{4}$$

we consider that the Θ sought-for is close to Θ_o, i.e. $\Theta = \Theta_o+\delta\Theta$, where $\delta\Theta << \Theta_o$. The solution of φ_o can sometimes be obtained in the analytical form, by using one of the approximate methods. The formal solution in the medium with the field of the Θ parameter is determined by

$$\phi=\phi_o+L_o^{-1}\delta L_{\theta\phi} \tag{5}$$

where $\delta L_\theta = L_\theta - L_o$
i.e. the complete model (3) taking account of (5) will be written as

$$U_n = H_n[\phi_o + L_\theta^{-1}\delta L_{\theta\phi}] + e_n \qquad (6)$$

In the model (6), the medium properties are included both in δL_θ, and in φ. Due to the presence of the dependence of f on δΘ in the determined part of the measurement model, the model appears to be nonlinear relative to δΘ, which results in great mathematical difficulties in obtaining a solution. But if δΘ is small, then if the condition is satisfied that

$$\frac{\|H_n L^{-1}\delta L_\theta(\phi-\phi_\theta)\|^2}{E\epsilon_n^2} \ll 1 \qquad (7)$$

(E is the operator of mathematical expectation), φ in (6) can be replaced by φ_o. From the physical point of view, the realization of the inequality (7) determines the adequacy of the model $u_n(\varphi)$ to the real measurement conditions; i.e. the model error resulting from the replacement of φ by φ_o in (6) is much smaller than the measurement error. It shoud be noted that due to the inequality

$$\|HL_o^{-1}\delta L_\theta(\phi-\phi_o)\| < \|HL_o^{-1}\delta L_\theta\| \, \|\phi-\phi_o\|$$

and the limitness of the norm of the difference of the fields of φ and φ_o from the physical considerations (physical fields do not possess infinite energy), as well as due to $\|\phi-\phi_o\| < C<\infty$ and the limitness of because of $\|HL_o^{-1}\delta L_\theta\|$ the compact operator H (which, being an integral convolution operator, is simultaneously responsible for cutting out the spatial-temporal interval detrmined by the registration condition, at δΘ tending to zero), the norm $\|HL_o^{-1}\delta L_\theta\|$ also tends to zero and the condition (7) is satisfied. Taking account of (7), we write down the model (6) as

$$u_n = H_n[\phi_o + L_o^{-1}\delta L_\theta \phi_o] + \tilde{\epsilon}_n \qquad (8)$$

The error $\tilde{\epsilon}_n$ includes both the random error ε_n, and the error relevant to the determined part of the model. These errors will be suppressed by the action of the operator H_n. We rewrite the model (8) introducing the belinear form:

$$u_n = \langle h_n|\phi_o\rangle_{\upsilon,T,\Omega} + \langle h_n|L_o^{-1}\delta L_\theta \phi_o\rangle_{\upsilon,T,\Omega} + \tilde{\epsilon}_n$$

By reducing the experimental data by the known value

$$<\xi|\eta>_{T,\upsilon,\Omega}=\int\limits_{\Omega}\int\limits_{\upsilon}\xi(\omega,x,t)*\eta(\omega,x,t)\,dx\,d\Omega$$

$U_n^o=<h_n|\phi_o>_{\upsilon,T,\Omega}$, we obtain

$$\tilde{u}_n=<h_n|L_o^{-1}\delta L_\Theta\phi_o>_{\upsilon,T,\Omega}+\tilde{\varepsilon}_n \qquad (9)$$

where $\tilde{u}_n=u_n-u_n^o$

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We distinguish in the disturbing operator δL_Θ the monotonous function v(Θ), relative to which the disturbing operator will be linear. We rewrite (9) as

$$\tilde{u}_n=<L_o^{-1}h_n^*|\delta L_\Theta\phi_o>_{\upsilon,T,\Omega}+\tilde{\varepsilon}_n=$$

$$=<G_o^*h_n|\frac{\delta}{\delta\upsilon}\delta L_\Theta|G_of>_{T,\Omega}|\upsilon(\delta\Theta)>_\upsilon+\tilde{\varepsilon}_n \qquad (10)$$

$$where\quad G_o=L_o^-1,\ \frac{\delta u}{\delta\upsilon}\delta L_\Theta:\ \frac{\delta u}{\delta\upsilon}=G_o^*h|\frac{\delta}{\delta\upsilon}\delta L_\Theta|\phi_o>_{T,\Omega} \qquad (10)$$

The integral kernel of the functional relative to v (δΘ) will take the name of tomography functional, and the designation will be introduced, viz.

$$\rho_n^\upsilon=<\phi_{out}|S^\upsilon|\phi_{in}> \qquad (11)$$

where $\varphi_{in}=\varphi_o$ is the incoming field in a known reference medium

$$\Theta_o,\ \phi_o:L_o\phi_o=f$$

$\phi_{out}:L_o^*\phi_{out}=h_n$- is the inversed outgoing field from the receiver,

$S^\upsilon=\frac{\delta}{\delta\upsilon}\delta L_\Theta$ is the operator of the interaction of the fields.

Taking accound of (11), we write the model (10) as

$$\tilde{u}=P_\upsilon+\tilde{\varepsilon}$$

where $u=[\tilde{u}_1,....,\tilde{u}_n,....,\tilde{u}_N]^T$, $\tilde{\varepsilon}=[\tilde{\varepsilon}_1,....,\tilde{\varepsilon}_n,....,\tilde{\varepsilon}_N]^T$

$$P=\begin{bmatrix} <\rho_{11}|\ldots<\rho_{1m}|\ldots<\rho_{1M}| \\ \cdots\cdots\cdots\cdots \\ <\rho_{n1}|\ldots<\rho_{nm}|\ldots<\rho_{nM}| \\ \cdots\cdots\cdots\cdots \\ <\rho_{N1}|\ldots<\rho_{Nm}|\ldots<\rho_{NM}| \end{bmatrix},\quad \upsilon=\begin{bmatrix} |\upsilon_1> \\ . \\ |\upsilon_m> \\ . \\ |\upsilon_M> \end{bmatrix}$$

The tomography functional ρ_n determines all of the elements of the spatial region in the sampling n of the experimental data. It should be noted that in the traditional ray tomography, the tomography functional is singular and is localized along the ray connection the source and the receiver; the weight along the ray is constant. Whereas in diffraction tomography, even under conditions of the applicability of the ray description of the fields φ_{in}, φ_{out}, every element of the volume of the region under investigation is linked with the receiver and the source by two ray trajectories, here every element of the spatial region has its own weight which is attributed to it and determined by the interaction of the fields φ_{in}, φ_{out}. It should be noted that the mathematical methods of computational tomography are based on the methods for the solution of the integral geometry problems, when data (projections) are given by integrals of parameter functions on manifolds of smaller dimensions (rays two-dimensional furfaces); whereas in diffraction tomography, the tomography functional support belongs to R^3. We note that the basic content of the tomography experiment is relevant to the overlapping of the tomography functional supports, i.e. information on the same element of the volume is contained in the whole set of measurements. The measurements are connected both with changes in the localization of the group of sources (φ_{in} in (11), and with the localization, orientation of the receiver, and the sampling time (f_{out} in (11) for dynamic fields.

The apriori presentations can be determined by either the probability or the determined technique, for instance, by giving a certain type of spatial symmetry. The determined form of giving the a priori information makes it possible to reduce the tomography functional to a space of smaller dimensions. If the medium is assumed a priori to be starti-homogeneous, the tomography functional support is one-dimensional; there the corresponding tomography functional is the Radon projection of the general tomography functional to the transversal direction. In the case of spherical symmetry, the parameter of the kernel of the tomography functional is the radial coordinate.

We shall now consider examples of the construction of the construction of the tomography functional.

1. Scalar wave equation

$$L_o=-\Delta+C_o^{-2}(x)\frac{\partial^2}{\partial t^2} \qquad (12)$$

$$L_\Theta=-\Delta+c^{-2}(x)\frac{\partial^2}{\partial t^2} \qquad (13)$$

$$\Theta_o=C_o(x),\quad \Theta=C(x),\quad x\in R^3,\quad \upsilon=\upsilon(x)=c_o{}^{-2}(1-\frac{c_o^2}{c^2}), \qquad (14)$$

$$S^\upsilon=\frac{\partial^2}{\partial t^2},\quad P=<\phi_{out}|\frac{\partial^2}{\partial t^2}|\phi_{in}>_T$$

In this case, the operator G^*:$G^*(t-t';\ x,x')=G(-(t-t');,x',x)$ determines the wave propagation from the receiver in the reverse time.

The tomography functional support in the homogeneous reference medium is concentrated in the paraboloid layer in the case of a plane incoming wave, and in the ellipsoid layer in the case of a spherical incoming wave (under condition of a point-type receiver).

2. The Lame equation in an isotropic infinite medium:

$$L_o\vec{\phi}=\rho_o\frac{\partial^2\vec{\phi}}{\partial t^2}[(\lambda_o+\mu_o)\vec{\nabla}\vec{\nabla}\cdot\vec{\phi}+\mu_o\Delta\vec{\phi}+\vec{\nabla}\lambda_o\vec{\nabla}\cdot\vec{\phi}+\vec{\nabla}\mu_o\times\vec{\nabla}\times\vec{\phi}+2(\nabla\mu_o\cdot\nabla)\vec{\phi}] \qquad (15)$$

$$L_\Theta\vec{\phi}=\rho\frac{\partial^2\vec{\phi}}{\partial t^2}[(\lambda+\mu)\vec{\nabla}\vec{\nabla}\cdot\vec{\phi}+\mu\Delta\vec{\phi}+\vec{\nabla}\lambda\vec{\nabla}\cdot\vec{\phi}+\vec{\nabla}\mu\times\vec{\nabla}\times\vec{\phi}+2(\nabla\mu\cdot\vec{\nabla})\vec{\phi}] \qquad (16)$$

where $\vec{\xi}\cdot\vec{\eta}$ is the scalar product of vectors $\vec{\xi}$ and $\vec{\eta}$

$$\Theta_o=\begin{vmatrix}\lambda_o(x)>\\ \mu_o(x)>\\ \rho_o(x)>\end{vmatrix},\quad \Theta=\begin{vmatrix}\lambda(x)>\\ \mu(x)>\\ \rho(x)>\end{vmatrix},\quad \upsilon(\delta\Theta)=\delta\Theta$$

Here, the structure of the operator P corresponding to an individual measurement is as follows:

$$<\rho^{\upsilon}|=\|<\rho^{\lambda}|<\rho^{\mu}|<\rho^{\rho}|\|$$

and the operator S has for the Lame equation can be presented in the explicity symmetric form.

$$S^{\lambda}:<\rho^{\lambda}(x)|=<\phi_{out}|S^{\lambda}|\phi_{in}>_T=-\vec{\nabla}\cdot\vec{\phi}_{out}*\vec{\nabla}\cdot\phi_{in} \tag{17}$$

$$S^{\mu}:<\rho^{\mu}(x)|=<\phi_{out}|S^{\mu}|\phi_{in}>_T=\vec{\nabla}\times\vec{\phi}_{out}*\vec{\nabla}\times\vec{\phi}_{in}-2\vec{\nabla}\vec{\phi}_{out}*\vec{\nabla}\vec{\phi}_{in} \tag{18}$$

$$S^{\rho}:<\rho^{\rho}(x)|=<\phi_{out}|S^{\rho}|\phi_{in}>_T=\frac{\partial}{\partial t}\vec{\phi}_{out}*\frac{\partial}{\partial t}\vec{\phi}_{in} \tag{19}$$

where the symbol * designates the convolution over time and summation over the indices of the spatial coordinates

$$\xi*\eta=\sum_i\sum_j\int d\tau\,\xi_{ij}(t_n-\tau)\eta_{ij}(\tau)$$

Let the variations of elastic parameters in an isotropic elastic medium be such that $\nabla\lambda,\nabla\mu$ are small enough. Then, under the condition that φ_{in} is a purely compressional wave, it is possible to introduce the function

$$u_{\rho\omega}=\rho_o[\frac{\lambda+2\mu}{\lambda_o+2\mu_o}-1]=\rho_o[\frac{u_\rho^2}{u_{\rho_o}^2}-1]$$

then $s^{\rho\omega}=\frac{\partial^2}{\partial t^2}$ (u_ρ-velocity of P-wave)

If φ_{in} is a purely transverse wave, then it is possible to introduce the function

$$u_{sw}=\rho_o[\frac{\mu}{\mu_o}-1]=\rho_o[\frac{v_s^2}{u_{so}^2}-1]$$

Then the interaction operator will be $S_s^{sw}=\frac{\partial^2}{\partial t^2}$, ($v_s$ - velocity of S-wave).

CONCLUSIONS

On the basic of a determined measurement model including per se the model propagation of the souding signal and the model of transformation of the signal field by the measurement channel, tomography functionals have been introduced. These functionals have been determined as the linear part of the mapping of the fields of the medium parameters sought for in discrete samples of the observation field.

The tomography functionals have the sense of the functions of the influence of various spatial regions of the medium on a particular measurement with a selected plan of the experiment.

The tomography functional norm (the influence function amplitude) at a fixed source and receiver is determined by the intensity of the interaction between the incoming field φ_{in} induced by the source and the inversed field φ_{out} emitted by the receiver. The operator of the interaction of the fields φ_{in} and φ_{out} is determined alone by the influence of a single variation of a concrete field of the medium parameters on the dynamics of the signal propagation and is independent of the choice of the reference medium model.

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