Space and Time in General Relativity

by

G. CONTOPOULOS

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(Received May 25, 1958)

The possibility of misundertandings regarding tbe geometry of a rotating disc shows the need for a consistent definition of time and space in the General Theory of Relativity. We define proper time and length by means of atomic elocks and rigid rods on the tangent euclidean spaee. Then the possible definitions of coordinate time and space are discussed and some eventual errors are pointed out. As an example we discuss tbe case of a rotating disc. It is found necessary, in general, to define $g_{\mu\nu}$ and tbe coordinate time a priori; then we can check these definitions by observations, namely by red-shift measurements. It is shown that the coordinate time cannot be taken arbitrarily; in fact it has the important physical property of being transmitted without change. This property is used for the synchronization of two distant clocks. The problem is more difficult in the case of an expanding Universe. It is seen that the Doppler and the gravitational red-shifts act together; they may be used as a general check of our hypotheses. Further the velocity is defined in three different ways; it is seen that in no case velocities greater than that of light exist. Finally an application of our difinitions on a rotating disc shows: 1) that the light rays are not 3-dimensional geodesics and 2) no shortening of the radius occurs.

1. INTRODUCTION

The great success of the mathematical treatment of the General Theory of Relativity has led to a tendency to avoid much discussion of the physical meaning of its contents. It is not even made always quite clear what is meant by the most elementary notions of time and space. According to Einstein (**b**, p. 81) in a gravitational field «it is not possible to obtain a reasonable definition of time with the aid of clocks which are arranged at rest with respect to the body of reference». Therefore «The scale and clock to some extent lose their preeminence» (Eddington **a**, p. 36). But clocks and scales are used, in connection with the Special Theory of Relativity, as an introduction to the General Theory. The most usual example of such a procedure is the case of a rotating disc (Einstein **a**, p. 58, Einstein **b**, p. 80-82, Möller p. 222 - 226, Gamow p. 67 - 70, Davidson p. 181, Couderc p. 107 - 110).

By this example it can be proved that the space on a rotating disc is not euclidean. However the usual proofs may give rise to misunderstandings. Is is mentioned e.g. that the measurements are considered «from the point of view of an observer in the system of inertia». But if we measure a number of rods laid down along the moving periphery, each of them will appear contracted to the observers at rest, «therefore» the whole periphery will appear s h o r t e r than $2\pi r$. If on the other hand we measure the length of the moving periphery with unit rods lying along it, we shall find it e q u a l to $2\pi r$, «because» the number of rods needed to complete the circumference does not change by its rotation.

The usual version of the problem is that the moving periphery is g r e a t e r than $2\pi r$, because the measuring rods on it will be contracted, «therefore» a greater number of them than $2\pi r$ will be needed to complete the circumference.

Eddington (**b**, p. 75) attributed the paradox of the rotating disc to the fact that the moving rods experience acceleration. It is not clear, however, how accelerations influence the behaviour of the moving rods. Möller and others (Möller p. 223, Einstein **a**, p. 59) assume that «the lengths of the rods are independent of the accelerations». It is necessary therefore, in order to give a correct answer to the problem of the rotating disc to make clear what is meant by length and time in the General Theory of Relativity.

2. PROPER TIME AND LENGTH

The whole Relativity is based on the invariance of the four-dimensional line element $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$. In order to define ds, it is supposed that we have a system of space coordinates x^1, x^2, x^3 and a time coordinate $t = \frac{x^4}{c}$. The coefficients $g_{\mu\nu}$ must fulfil the conditions $g_{11}>0$, $\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0$, $\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} > 0$, $g_{44}<0$ in order that the above quadratic form should be positive definite for $dx^4 = 0$ and negative for $dx^1 = dx^2 = dx^3 = 0$. Such conditions are necessary if a basic distinction between space and time is to be assumed.

In a given model $g_{\mu\nu}$ are given functions of x^i ; but ds^2 remains invariant by any change of coordinates $x^i = x^i (x^{j'})$, i.e. $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\varkappa\lambda}' dx^{\varkappa'} dx^{\lambda'}$. In this model the motions of the material particles and of the light rays are represented by geodesics in the four-dimensional space (not in the three-dimensional space in general). The form of these geodesics does not depend on the choice of coordinates. Let us take geodetic coordinates at a given point - event P (x^1, x^2, x^3, x^4). Then the derivatives of $g_{\mu\nu}$ at this point are zero (von Laue p. 73).

Therefore we have in first approximation an euclidean space. By a suitable transformation of coordinates the line element assumes the form $ds^2 = d\sigma^2 - c^2dt^2$ where $d\sigma^2 = dx'^2 + dx''^2 + dx'''^2$ in the new system. This euclidean space may be called tangent to the given space at the point-event P, in the same sense as a plane is tangent to a given surface at one of its points. But this euclidean space contains an infinity of inertial systems* : every two such systems are connected by a Lorentz transformation, but for all of them we have $ds^2 = d\sigma^2 - c^2dt^2 = d\sigma'^2 - c^2dt'^2$. If M represents a material point which at the given time $t = \frac{x^4}{c}$ is at the point (x^1, x^2, x^3), then there is one system S with axes going permanently through M, i.e. having as time axis the world line of M. It is evident that only one such system exists ; the velocity of M with respect to S is zero.

In this system it is again $ds^2 = dl^2 - c^2d\tau^2$ and now $d\tau$ is the proper time and dl the proper length in the neigbbourhood of M. If $dl = 0$, then $ds = icd\tau$; $d\tau$ measures the proper time in M, i.e. the time that shows an atomic clock there, e.g. an ammonia clock, (the second is then d e f i n e d as a given number of vibrations), or else a radioactive clock, with constant half - life.

It is said sometimes that such clocks are not influenced by acceleration. Papapetrou (p. 116) e.g. introduces the «hypothesis» that there exist «normal» clocks, whose proper period is not changed by a gravitational field, and that the atoms are such «normal» clocks. It is easily seen however that this is not an hypothesis at all ; in fact the proper time is d e f i n e d as that given by atomic clocks.

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* By «system» we mean a three - dimensional body of reference ; the space axes may have any directions in it.

Further it is usually assumed that in order that a clock should show the proper time, it should fall freely in a gravitational field, describing a geodesic. This is a consequence of the principle of equivalence, because for a system moving along with the clock no strain or acceleration is present. However such is not the case e.g. for the clocks on a rotating disc. Further, as Papapetrou (ibid.) has noticed, a pendulum clock does not give any time at all, if it falls freely, because then it does not oscillate. The only escape from such dilemmas is again to define proper time as the time that is given by atomic clocks, in all cases.

If we set $d\tau = 0$ in the equation above, then $ds = dl$; ds measures the proper length in the neighbourhood of the point M. It is supposed that rigid rods do not change their length; or better, that they give, by definition the proper length. Now it is *verified experimentally* that light rays are represented by the null geodesics $ds = 0$, $\delta\int ds = 0$. Hence $\frac{dl}{d\tau} = c$, i.e. the proper velocity of light is constant.

This is to be taken as an experimental fact*, which has been verified in connection with the Special Theory of Relativity. It is assumed to be valid in every tangent euclidean space; i.e. it is assumed that in every tangent euclidean space Special Relativity holds.

3. COORDINATE SPACE AND TIME

We come now to the problem how the form of ds is found and what are the physical meanings of the coordinates x^1, x^2, x^3, x^4. The space coordinates x^1, x^2, x^3 can be easily defined. They correspond to arbitary markings in a three-dimensional space. If we have rigid rods laid down along definite directions, whatever markings on these rods may give a system of coordinates x^1, x^2, x^3. The difficult point is to define t, which is called the «coordinate time». At first sight t may be taken arbitrarily (Einstein **b**, p. 99) e.g. it is given by the readings of any physical phenomenon, which is used as a clock (Contopoulos **a**, p. 202 - 204). The law of the physical pheno-

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* On the contrary Milne (p. 27 - 28) defines the length by this equation, i.e. he does not accept the rigid rod as a primary notion or as a necessary tool.

menon is supposed to be known, therefore t can be deduced from the proper time τ by an arbitrary, but given transformation of coordinates, involving eventually the space coordinates x^1, x^2, x^3 also. Sometimes t is defined by the readings of a clock in another system (e.g. for a rotating disc, the readings of a non rotating clock); then again there is a special (but arbitrary) transformation which connects our readings of proper time τ with the readings of this clock, which gives the coordinate time t. However, as we shall see, t has a certain physical meaning, therefore it cannot be taken arbitrarily.

When the x^i are defined, the form of ds^2 may be found in the following two ways : a) It is supposed that the form of ds is known in a given system and from it we derive its form in another system, moving in any g i v e n way with respect to the first, by a suitable transformation of coordinates. We note that the world-line of M is defined by $dx^1 = dx^2 = dx^3 = 0$; the time axis is not changed when we move from the (x^1, x^2, x^3, x^4) to the (l, τ) system. Therefore dl is a function of (x^1, x^2, x^3) only ; in other words dl is vertical to the time-axis (von Laue p. 142) Then it follows (either from the condition that dl is independent from x^4, or from the verticality condition) that :

$$c d\tau = \sqrt{-g_{44}}\left(dx^4 + \frac{1}{g_{44}}\sum_{\iota=1}^{3} g_{\iota 4} dx^\iota\right)$$

If we set $\gamma_\iota = \frac{g_{\iota 4}}{\sqrt{-g_{44}}}$ and $\gamma_{\iota\varkappa} = g_{\iota\varkappa} + \gamma_\iota\gamma_\varkappa$, we find :

$$c d\tau = (c\sqrt{-g_{44}}\, dt - \gamma_\iota\, dx^\iota) \text{ and}$$

$dl^2 = \frac{1}{g_{44}}\sum_{\iota,\varkappa=1}^{3} (g_{\iota\varkappa}g_{44} - g_{\iota 4}g_{\varkappa 4})\, dx^\iota dx^\varkappa = \gamma_{\iota\varkappa} dx^\iota dx^\varkappa$, where ι, $\varkappa$, vary from 1 to 3*. It is then verified that $dl^2 - c^2 d\tau^2 = ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$.

As an example, let us take the case of the rotating disc. In the inertial system of the center of the disc we have $ds^2 = dX^{1 2} + dX^{2 2} + dX^{3 2} - c^2 dT^2$, i.e. our four-dimensional space is euclidean. We

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* The same conclusion is found by Möller by another method. But his formulae are derived for the special case when $\breve{A}_4^\iota = \frac{\partial X^\iota}{\partial x^4} = 0$. Möller uses these formulae in the case of a rotating disc (p. 240 - 241); but there his proof is not valid, because $\breve{A}_4^1 = -\frac{r\omega}{c}\sin(\vartheta + \omega t)$ and $\breve{A}_4^2 = \frac{r\omega}{c}\cos(\vartheta + \omega t)$ i.e. in general $\breve{A}_4^1 \neq 0$, $\breve{A}_4^2 \neq 0$.

assume that these coordinates are connected to the cylindrical coordinates (r, ϑ, z) on the moving disc by the usual non-relativistic equations $x^1 = r \cos(\vartheta + \omega t)$, $x^2 = r \sin(\vartheta + \omega t)$,

$$x^3 = z, \quad T = t = \frac{x^4}{c}.$$

Then it is found that $ds^2 = dr^2 + r^2 d\vartheta^2 + dz^2 + 2\omega r^2 d\vartheta dt -$ $- (c^2 - r^2\omega^2)\, dt^2$ i.e. $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = 1$,

$$g_{12} = g_{13} = g_{14} = g_{23} = g_{34} = 0, \quad g_{24} = \frac{\omega r^2}{c}, \; g_{44} = -\left(1 - \frac{r^2\omega^2}{c^2}\right).$$

From this we find for the proper time and length the expressions

$$d\tau = \sqrt{-g_{44}}\left(dt + \frac{g_{24} d\vartheta}{c\, g_{44}}\right) = \sqrt{1 - \frac{r^2\omega^2}{c^2}}\, dt - \frac{r^2\omega d\vartheta}{c^2\sqrt{1 - \frac{r^2\omega^2}{c^2}}} \text{ and } dl^2 =$$

$$= dr^2 + \frac{r^2 d\vartheta^2}{1 - \frac{\omega^2 r^2}{c^2}} + dz^2.$$ We come to the same formulae if we take a system moving with the velocity of a given point (r_0, ϑ_0, z_0) at a given moment t_0 in its rotational motion around the center ; in this system the length and time are evidently proper length and proper time*.

The above space element dl, defined in polar coordinates r, ϑ, z is evidently not euclidean. From this it follows that the length of the periphery is $\frac{2\pi r}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}$, i.e. it is greater than $2\pi r$.

The same conclusion can be drawn from the following considerations. If dl_0 is the length of a rod on the moving periphery, its length measured by an observer at rest will be $dl_1 = dl_0\sqrt{1 - \frac{r^2\omega^2}{c^2}}$. Hence $dl_0 = \frac{dl_1}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}$ and as the integral of dl_1 is $l_1 = 2\pi r$, it is found $l_0 = \frac{2\pi r}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}$.

* Although there is a general agreement on the non-euclidean character of the space of a rotating disc, some authors have expressed different views. Glaser (p. 75) e.g. stresses the fact that the four-dimensional space is euclidean. This is true, but it is nevertheless possible that a curved three-dimensional space belongs to a flat four-dimensional space.

We cannot assume that $l_0 = 2\pi r$, so that $l_1 = 2\pi r \sqrt{1-\frac{r^2\omega^2}{c^2}}$, because l is not known a priori; therefore the possibilities that the periphery should be smaller or equal to $2\pi r$ are excluded. The only difficulty arises from the fact that we cannot visualise how a greater number of measuring rods is needed to cover the moving periphery than the non moving one. But when we measure dl_1 from outside, we have everywhere the same time t, while the proper time τ on the moving periphery is different from point to poiut. However, τ must be the same for the observers on the rotating disc when they measure a length dl_0. In fact, for $dt = 0$ it is $d\tau = -\frac{r^2\omega d\vartheta}{c^2\sqrt{1-\frac{r^2\omega^2}{c^2}}}$ and

$dl_1^2 = dl_0^2 - c^2 d\tau^2$, and as $dl_1 = r d\vartheta$, we find $dl_0^2 = dl_1^2 + \frac{r^2\omega^2 dl_1^2}{c^2\left(1-\frac{r^2\omega^2}{c^2}\right)} =$

$= \frac{dl_1^2}{\left(1-\frac{r^2\omega^2}{c^2}\right)}$ as it was expected. In other words l_0 is greater than l_1; it should be equal to it only if $d\tau = 0$, which is not the case if $dt = 0$.

Iu the above example it is to be noted that although the space of the disc is not euclidean, the four - dimensional space - time is euclidean; this cau be proved by returuing to the initial coordinates X^1, X^2, X^3, T. This remark has some important consequeuces, as we shall see later.

b) Another method by which we can find the form of ds^2 is proposed by Möller (p. 237 - 240); it is based on a straightforward experimental determination of the $g_{\mu\nu}$.

If we have coordinate clocks at rest in every point of the space, giving the coordinate time t, we can find g_{44} from the equation $d\tau = \sqrt{-g_{44}}\, dt$, if a proper (atomic) clock is also given. Further γ_ι and $\gamma_{\iota\kappa}$ are found by measuring the velocity of light in different directions (in t-time). But, as we shall see, t cannot be taken arbitrarily; g_{44} is given from red - shift measurements, hence t is also defined. Therefore we cannot find g_{44} by the above method. However the space coordinates are arbitrary, i.e. the coefficients γ_ι, $\gamma_{\iota\kappa}$ may be found, in principle, by Möller's method.

Another difficulty of this method is that it is impossible to make measurements of the coordinates in all the points of the space. Not only our measurements are discontinuous, but they do not cover, even sparsely, but a small part of the whole space. In cases of cosmological siguificance our measurements cannot give us anything about the form of the world - geometry, as they are coufined only to the earth. Therefore some a priori assumption concerning the values of $g_{\mu\nu}$ is necessary. We shall see this method presently.

4. A PRIORI SOLUTIONS

In all the applications of the General Theory of Relativity the $g_{\mu\nu}$ are defined a priori, by general considerations regarding the potential fields pervading the space. E.g. $g_{\mu\nu}$ are found by solving the Einstein equation of gravitation $G_{\mu\nu} = 0$. In other cases, as in the case of the rotating disc, the form of the space time is supposed known for a special system, by general considerations also, or just by an a priori assumption ; then the form of $g_{\mu\nu}$ in another system is found by a simple change of coordinates.

The coordinates are supposed to be given by special readings of measuring rods and specially graduated clocks. Mc Vittie (p. 35) uses a method of defining the coordinate readings of the clocks and the space coordinates, by using proper time and lengths. It is fouud (by changing Mc Vittie's notations in order to conform to ours) :

$$s_1 = \int_0^{s_1} ds = c\int_0^{t_1} \sqrt{g_{44}(0,0,0,t)}\, dt = ic\tau_1, \quad \text{where} \quad x^1 = x^2 = x^3 = 0.$$

This equation gives the increase of coordinate time t from the increase of proper time τ, if the clock is at rest in the system x^1, x^2, x^3. Further for two events $(0,0,0,t_0)$ and $(x^1_1, 0,0, t_0)$ we have:

$$1 = \int_0^{x^1_1} \sqrt{g_{11}(x^1,0,0,t_0)}\, dx^1.$$

Now 1 is the proper length along the coordinate line x^1 ; it is supposed that this line has been defined in some way (e.g. by the track of a light ray) but it is not graduated). Then our equation gives the values of x^1 for different points of this line. By similar equations

$$1 = \int_0^{x^2_1} \sqrt{g_{22}(x^1_1, x^2, 0, t_0)}\, dx^2$$

e.t.c. the other two coordinates are

defined. Mc Vittie's method is right, but not quite general. In the case that the $g_{\iota 4}$ are not all zero, the first equation gives again the time t_4 for a stationary clock. In the other equations we must use $\gamma_{\iota\kappa}$ instead of $g_{\iota\kappa}$. E. g. $1 = \int_0^{x^1_1} \sqrt{\gamma_{\iota\kappa}(x^1, 0, 0, t_0)}\, dx^1$ e.t.c.

By this method x^1, x^2, x^3 are defined, when the $g_{\mu\nu}$ are known. In practice, however, x^1, x^2, x^3 are somehow defined a priori, so that the above equations can only check this definition ; i.e. our experiments give simply a verification or a disproof of our assumptions regarding the nature of x^1, x^2, x^3 and the forms of the $g_{\mu\nu}$.

A more difficult problem arises, when the $g_{\mu\nu}$ are not completely defined. E.g. in the case of a gravitational field in a non void space it is $G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G = kT_{\mu\nu}$. In order to find $T_{\mu\nu}$ we must know the distribution of matter in space; but this is not known until some space frame is laid down, i.e. until $g_{\mu\nu}$ is known. This difficulty is a basic one (Mc Vittie p. 42) and does not permit a definition of $g_{\mu\nu}$ exclusively a priori. However, we can use an approximate form of $g_{\mu\nu}$ in determining $T_{\mu\nu}$ and then we may find better values of $g_{\mu\nu}$ by a method of successive approximations. In practice, we use a model of the Universe where $T_{\mu\nu}$ is known (e.g. the Universe is supposed homogeneous) and our conclusions are then compared with the observations.

5. SYNCHRONIZATION AT A DISTANCE

When we measure the time t needed by a particle or a light ray to describe a certain distance (in a given system), it is tacitly assumed that we have a means of synchronizing the clocks along the path, so that they measure t h e s a m e time everywhere. Of course a proper clock moving together with the particle gives some proper time intervals. E.g. in the case of a light ray, these time intervals are always zero. However, proper time is not suitable for synchronizing distant clocks. This is because synchronization by proper time is not consistent, i.e. two distant proper clocks which are once found to be synchronous are no more synchronous, in general, after some time*. In other words if two light rays start from a given place with proper time difference $d\tau$, they do not arrive, in general, with the same proper time difference at another place. In a static field (i.e. when $g_{\mu\nu}$ are independant of t) only the coordinate time t possesses, in fact, this important property. The coordinate time intervals dt are transmitted unchanged by the light rays (von Lane p. 144 and 188, Tolman p. 222). Actually, t is defined in such a way, so as to be transmitted invariable (Eddington **a**, p. 92).

This difference between proper and coordinate time can be best illustrated in the case of the Einstein red-shift of the light rays from the sun. The proper time interval (proper period) of a vibration of an atom in the sun (at rest relatively to the snn) is $d\tau = \sqrt{1-\frac{2\mu}{r}}\,dt$. It is known that dt remains unchanged as the light is transmitted, while $d\tau$ changes. At the distance of the earth (practically r=infinite) dt is the same, therefore $d\tau$ becomes : $d\tau_1=dt$, while the proper time on earth is $d\tau_e=dt_e$. The intervals $d\tau$ of the vibrations of two similar atoms on the sun and on the earth are equal by definition (i.e. two atoms are regarded as similar, if the $d\tau$ of their vibrations are equal). Therefore $d\tau = d\tau_1\sqrt{1-\frac{2\mu}{r}}$ or $d\tau_e = d\tau_1\sqrt{1-\frac{2\mu}{r}}$, i.e. the proper period of the arriving light $d\tau_1$ is longer than the proper period of the corresponding light of the earth. One could perhaps think that the same should happen if instead of t another coordinate time t′ and a corresponding form of g_{44} were used. But then the red-shift should be different. Therefore, the red-shift observed in the light coming from a point where g_{44} has a different value than on earth defines the coordinate t to be used, thus defining also g_{44}. In general, the period of a light pulse $d\tau$ at the starting point P_1 is given by $ds^2 = -c^2d\tau^2 = -c^2g_{44}(P_1)dt^2$; on arrival of the light pulse to the point P_2, ds^2 becomes $ds_1^2 = -c^2d\tau_1^2 = -c^2g_{44}(P_2)dt^2$, while the light pulse of an atom at the point P_2 gives again ds^2. Therefore $ds = ds_1\sqrt{\frac{g_{44}(P_1)}{g_{44}(P_2)}}$ or $d\tau = d\tau_1\sqrt{\frac{g_{44}(P_1)}{g_{44}(P_2)}}$ and the red-shift is

* Mc Vittie (p. 37 - 38) thinks that we can synchronize two distant proper clocks by moving the first with infinitesimal velocity up to the place of the other. We shall see, however, that proper time intervals are not preserved. Mc Vittie assumes that when $v \to 0$, then vt tends also to zero; in fact, then $t \to \infty$, so that vt remains finite (= x′).

given by $\frac{1+dl}{l} = \sqrt{\frac{g_{44}(P_1)}{g_{44}(P_2)}}$; from this equation g_{44} can be derived. In general, however, g_{44} is known a priori and the law of red-shift gives only a verification of the general considerations which lead to the given value of g_{44} (g_{44} is given as a function of the coordinates x^1, x^2, x^3, which are supposed to be defined in some arbitrary way, so that every emission and reception point should have definite coordinates x^i_1, x^i_2). In the case where only the f o r m of g_{44} is known a priori, x^1, x^2, x^3 must be d e f i n e d in such a way so that g_{44} should have at the points P_1 and P_2 values that are consistent with the red-shift measurements.

However, x^1, x^2, x^3 are in practice assumed to be known a priori (e.g. the value of the sun's radius r in the case of the Einstein red-shift is known), so that the red-shift measurements give again only a verification or disproof of our a priori assumptions.

A more difficult case occurs when $g_{\mu\nu}$ are variable with time, e.g. in an expanding Universe. Then dt is not transmitted invariable, in general, and the change of ds is not always the same for all t. In the general case when $g_{\mu\nu}$ are functions of the space coordinates and of the time t, it is not generally possible to find experimentally the $g_{\mu\nu}$ by using arbitrary coordinates or inversely to define the coordinates by using arbitrary $g_{\mu\nu}$. Therefore, we use a priori forms of ds^2, refering to known coordinates, and try to verify through experiments the consequences of these assumptions.

For a light ray we have $ds = 0$ and $\delta \int ds = 0$. Therefore, we can find x^1, x^2, x^3 along the path as functions of t and of the initial conditions x^1_1, x^2_1, x^3_1, t_1. Hence we find $x^i_2 = x^i_2 (x^i_1, t_1, t_2)$. From these equations we can find t_2 by giving to x^1, x^2, x^3 some special values, i.e. for specific points x^i_2 of the space. In the case of the line element of an isotropic Universe (Tolman p. 364 - 389; a small change in the notations is used) we have :

$$ds^2 = \frac{e^{g(t)}}{\left(1+\frac{r^2}{4R_0^2}\right)^2} (dr^2 + r^2 d\vartheta^2 + z^2 \sin^2\vartheta d\varphi^2) - c^2 dt^2.$$

Then we have the following equation for a radial light ray

$$\int_{t_1}^{t_2} e^{-\frac{1}{2}g(t)} dt = \int_0^r \frac{dr}{c\left(1+\frac{r^2}{4R_0^2}\right)},$$ which d e f i n e s t_2 when r is

given. In all cases where x^1_i, x^2_i, x^3_i $(i = 1, 2)$ are kept constant, t_2 is a function of t_1, i.e. $t_2 = f(t_1)$ and $dt_2 = \frac{df}{dt_1} dt_1$. Therefore, if the coordinate period of a light pulse in coordinate time at a given point x^i_1 is dt_1, the corresponding coordinate period at x^i_2 is dt_2; the corresponding proper time intervals for clocks at rest on the points x^i_1 and x^i_2 are $d\tau_1 = \sqrt{-g_{44}(P_1)}\, dt_1$ and $d\tau_2 = \sqrt{-g_{44}(P_2)}\, dt_2$. However the proper period of the vibration of an atom at x^i_2 is equal by definition to the proper period of a similar atom at x^i_1. Therefore the proper periods of the arriving and local light pulses have a ratio $\frac{d\tau_a}{d\tau_e} = \sqrt{\frac{g_{44}(P_2)}{g_{44}(P_1)}}\, \frac{dt_2}{dt_1}$ and the wave lengths have a ratio

$\frac{l + dl}{l} = \sqrt{\frac{g_{44}(P_2)}{g_{44}(P_1)}}\, \frac{dt_2}{dt_1}$. This red-shift is called the relativistic Doppler shift. It is evident that it is of the same nature as the Einstein red-shift discussed above. Both are due to the form of the g_{44} with tbe only difference that in the usual Einstein red-shift the g_{44} is supposed independent of the time.

In the above formula $\frac{l + dl}{l}$ is a function of x^i_1, x^i_2 and t_1 or t_2; t_1 and t_2 are connected by the formula $t_1 = f(t_2)$, which contains also, in general, x^i_1 and x^i_2. Further, the equations of the light ray $ds = 0$, $\delta \int ds = 0$ give two more relations between x^i_1 and x^i_2. Therefore, we have four equations which define x^i_2 and t_2 as functions of x^i_1, t_1. Conversely, we may find the values of t_1 and x^i_1 from the known values of t_2 and x^i_2 at the point of observation (x^i_2 may be taken arbitrarily). But in practice x^i_1, x^i_2 are supposed known, so that the observed red-shift is used only to verify our assumptions concerning the form of $g_{\mu\nu}$.

Tolman (p. 289) used a more general formula assuming that both the source and the observer are moving with respect to the coordinate system x^1, x^2, x^3.

Then it is found $\frac{l + dl}{l} = \frac{df(t_1)}{dt_1} \frac{\sqrt{\left(g_{11}\left(\frac{dx^1}{dt}\right)^2 + \cdots + g_{44}\right)_2}}{\sqrt{\left(g_{11}\left(\frac{dx^1}{dt}\right)^2 + \cdots + g_{44}\right)_1}}$. In the special case of an isotropic Universe and an observer permanently located at the origin of coordinates it is found (Tolman p. 390)

$$\frac{1+dl}{l} = \frac{e^{\frac{1}{2}(g_2 - g_1)}}{\sqrt{1-\frac{u^2}{c^2}}}\left(1+\frac{u_r}{c}\right)$$ where u is the velocity of the source with respect to the system of coordinates r, ϑ, φ (u_r is the radial velocity). If $u = 0$, the red-shift is not zero; there is again a Doppler shift due to the expansion of the space. It is evident that the red-shift changes with time. But its change is insignificant for small time intervals (small in the cosmic time-scale of course); thus no such effect has yet been observed, because our observations cover only time intervals of the order of a few decades at most. It is to be noted that in all cases the ratio $\frac{1+dl}{l}$ is independent of l, i.e. the fractional red-shift $\frac{dl}{l}$ is the same over the whole spectrum. This conclusion has recently received an important verification by radio astronomical observations of the red-shift of the Cygnus A radio-source (Lilley and McClain, Minkowski and Wilson). These observations show that the fractional red-shift is constant for a wave length range from 500,000 to 1, thus supporting the view that the observed red-shift of the distant galaxies is really a relativistic Doppler shift.

Tolman claims further (p. 289) that the fractional red-shift $\frac{dl}{l}$ is an absolute quantity in the sense that it is independent of the particular coordinates used to describe the effect. We know, however, that this is not the case. Suppose that we have the simple case of a light ray in the field of the sun, which has been discussed above. For a radial ray we have : $ds^2 = \frac{dr^2}{1-\frac{2\mu}{c^2 r}} - c^2\left(1-\frac{2\mu}{c^2 r}\right)dt^2 = 0$, or $\frac{dr}{dt} = c\left(1-\frac{2\mu}{c^2 r}\right)$; thus $\frac{dr}{dt}$ is independent of t. Therefore dt is transmitted unchanged ; i.e. $dt_2 = dt_1$. Then $\frac{1+dl}{l} = \frac{1}{\sqrt{1-\frac{2\mu}{c^2 r}}}$, if the distance of the earth is supposed very great. But if we use, instead of the above coordinate time, a new coordinate time t' equal to the proper time τ, the line element for a radial ray becomes $ds^2 = \frac{dr^2}{1-\frac{2\mu}{c^2 r}} - c^2 dt'^2$; hence $\frac{dr}{dt'} = c\sqrt{1-\frac{2\mu}{c^2 r}}$, which is again independent of t′. Therefore, it should be : $dt_2' = dt_1'$ and $\frac{l + dl}{l} = 1$. This result, which is not compatible with the observations, is different from the above, because the proper time τ is not transmitted nnchanged like the coordinate time t. In general the calculated $\frac{l + dl}{l}$ depends on the time coordinate used. But as $\frac{l + dl}{l}$ is found by observations, the red-shift law may be nsed as a check of our hypotheses concerning the form of g_{44} and the time that is used as a coordinate time.

6. VELOCITY

The most natural way to define the velocity of a moving particle shonld be to divide the proper length by the proper time needed to describe this length. This method may hc used only for infinitesimal distances, because the proper time is not transmitted invariable. Therefore snch a velocity cannot be used in integrations; e.g. if v is constant we cannot deduce that $l = v\tau$. (On the other hand, we cannot use the time of a proper clock moving along with the moving body, becanse such a time is always zero in the case of a light ray).

A better definition of velocity is given by the coordinate velocity, which is the ratio of space and time coordinates, with projections $\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}$. Such a velocity is nsed in most cases of integration. But with such a definition the velocity of light is not constant. In the case of the field of the sun, where $ds^2 = \frac{dr^2}{\left(1 - \frac{2\mu}{c^2 r}\right)} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - c^2\left(1 - \frac{2\mu}{c^2 r}\right)dt^2$, the velocity of light is given by the equation $\frac{\left(\frac{dr}{dt}\right)^2}{\left(1 - \frac{2\mu}{c^2 r}\right)} + r^2\left(\frac{d\vartheta}{dt}\right)^2 + r^2\sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2 = c^2\left(1 - \frac{2\mu}{c^2 r}\right)$; i.e. the velocity of light can be represented by a vector forming an oblate ellipsoid with radial and tranverse axes $c\left(1 - \frac{2\mu}{c^2 r}\right)$ and $c\sqrt{\left(1 - \frac{2\mu}{c^2 r}\right)}$ respectively (Eddington **a**, p. 93).

More often a third definition of velocity is used, which is the ratio of the proper length and the coordinate time (von Lane p. 144). With this definition, the velocity of light is again variable. It may be given as $V = \frac{dl}{dt} = c\sqrt{-g_{44}}$. This velocity may also be used in integration, because proper lengths can be added. It is necessary, however, in order to avoid mistakes and misunderstandings, to determine in every case what kind of velocity is used. E.g. it is said sometimes that in General Relativity there are velocities greater than the velocity of light. This is wrong, except in the sense that the number giving the velocity may be greater than c, which is trivial. If the velocity is defined by $v = \frac{dl}{dt}$, it is always $v < V$, where V is the velocity of the light. This is because for a moving particle $ds^2 = dl^2 + c^2 g_{44} dt^2 = -c^2 d\tau^2 < 0$, where $d\tau$ is the proper time of a clock moving with the particle, i.e. $v^2 < -c^2 g_{44}$, while for a light ray $ds^2 = 0$, i.e. $V^2 = -c^2 g_{44}$. The same is the case for the coordinate velocity. If $dx^2 = dx^3 = 0$, it is $ds^2 = g_{11} dx^{12} + 2cg_{14} dx^1 dt + c^2 g_{44} dt^2 < 0$ for a particle, while for a light ray it is $ds^2 = g_{11} dx^{12} + 2cg_{14} dx^1 dt + c^2 g_{44} dt^2 = 0$. This last equation has two roots for $\frac{dx^1}{dt}$, the one positive and the other negative, because $g_{11} > 0$ and $g_{44} < 0$. The positive root is the velocity of the light. In the case of the particle $ds^2 < 0$, therefore in this case $\frac{dx^1}{dt}$ lies between the two roots of $ds^2 = 0$, i.e. $\frac{dx^1}{dt}$ is smaller than the velocity of light.

We may thus say that in every case and by all definitions of velocity, the velocity of light is always the greatest possible one.

7. APPLICATION TO THE ROTATING DISC

Let us apply our general considerations to a special problem. We shall discuss the form of the orbits of the light rays on a rotating disc. These are defined as straight lines in the four-dimensional space $ds^2 = dX'^2 + dX''^2 + dX'''^2 - c^2 dT'^2$. If we use polar coordinates, $ds^2 = dr^2 + r^2 d\vartheta + dz^2 + 2\omega r^2 d\vartheta dt - (c^2 - r^2\omega^2) dt^2$. For a light ray we have $\left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\vartheta}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2 + 2\omega r^2 \frac{d\vartheta}{d\lambda}\frac{dt}{d\lambda} - (c^2 - r^2\omega^2)\left(\frac{dt}{d\lambda}\right)^2 = 0$, and

$$\delta\int\left[\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{d\vartheta}{d\lambda}\right)^2+\left(\frac{dz}{d\lambda}\right)^2+2\omega r^2\frac{d\vartheta}{d\lambda}\frac{dt}{d\lambda}-(c^2-r^2\omega^2)\left(\frac{dt}{d\lambda}\right)^2\right]d\lambda=o$$

The Euler equations for ϑ and t give then :

$$2r^2\frac{d\vartheta}{d\lambda}+2\omega r^2\frac{dt}{d\lambda}=2c_1\ ,\quad 2\omega r^2\frac{d\vartheta}{d\lambda}-2\,(c^2-r^2\omega^2)\frac{dt}{d\lambda}=2c_2\omega$$

hence : $\frac{c^2}{\omega}\frac{dt}{d\lambda}=c_1-c_2$, or $\frac{dt}{d\lambda}=u=\text{const.}$

Then $\frac{d\vartheta}{d\lambda}=\frac{c_1}{r^2}-u\omega$. It is also $\frac{dz}{d\lambda}=q$ and for a plane motion q=0.

Finally : $\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{c_1}{r^2}-u\omega\right)^2+2\omega r^2 u\left(\frac{c_1}{r^2}-u\omega\right)-(c^2-r^2\omega^2)u^2=0$

or : $\left(\frac{dr}{d\lambda}\right)^2+\frac{c_1^2}{r^2}-c^2u^2=0.$

Therefore $\frac{dr}{d\lambda}=\pm\sqrt{c^2u^2-\frac{c_1^2}{r^2}}$ and $\frac{dr}{d\vartheta}=\pm\frac{r^2\sqrt{c^2u^2-\frac{c_1^2}{r^2}}}{(c_1-u\omega r^2)}$.

The orbits are similar in form with the geodesics of the three-dimensional space of the rotating disc (Möller p. 241), but they do not coincide with them, as thinks wrongly Gamow (p. 69). In fact, the light rays are straight lines relatively to an inertial system; relatively to the rotating disc they are bent so that their ends lag always behind, opposite to the sense of rotation. This should be attributed, from the point of view of a rotating observer, to a combination of centrifugal and Coriolis forces on the moving ray. The corresponding calculations should take Coriolis forces into account. In particular they should take into account the motion of the ray (or particle) with respect to the disk in deriving ds, while the proper space calculated above $dl^2=dr^2+\frac{r^2d\vartheta^2}{1-\frac{r^2\omega^2}{c^2}}+dz^2$ has been found only for a point non-moving with respect to the disc.

We shall mention finally another example of a questionable application of general considerations to a special problem. Eddington (**a**, p. 112 - 113) claims that the radius of a rotating disc is subject to a contraction which is «one quarter of that predicted by a crude application of the FitzGerald formula to the circumference». But all our calculations concerning the geometry of a rotating disc are based on the invariance of the length of the radius ; this is shown clearly in the form of the line element of the rotating disc. Eddington's statement derives from the assumption that the particle density in the rotating disc is equal to that of a non-rotating disc. Then in order that the total number of particles on the disc should be constant, the radius must shrink, as the circumference is greater than $2\pi r$. But the particle-density changes by just the amount needed to counteract the change of the periphery ; in fact the number of particles on a given space-element, limited by some definite boundary, is the same whether the disc is rotating or not (the boundary is the same materially, but eventually deformed by rotation). Therefore the number of particles remains the same without any contraction of the radius.

These examples show the possibility of deriving wrong results from the General Theory of Relativity. This theory, however, if correctly applied gives valuable information about the behaviour of clocks and scales and the motion of particles or light. Such an example is the celebrated clock paradox, which is fully accounted for by the General Relativity (Möller p. 258 - 263, Contopoulos **b**, p. 33 - 36). But in order to draw right conclusions it is necessary to clarify, as far as possible, the meanings of time and space in General Relativity.

I wish to thank Prof. O. Pylarinos for useful discussions and suggestions in preparing this paper.

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