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CONDITIONS FOR RECIPROCALITY OF ω - PERIODIC LINEAR DIFFERENTIAL SYSTEMS

by

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Abstract: *J. Hale in his book [2, p. 132] proves that sufficient conditions, in order an ω-periodic linear system of differential equations $\dot{x} = A(t)x$ to be reciprocal, are (a) $DA(t) + A^T(t)D = 0$ or (b) $DA(t) + A(-t)D = 0$, where D is a constant nonsingular matrix. The purpose of this paper is firstly to derive from (a) stability properties of the system and to provide a way of determination of the matrix D and secondly to generalize conditions (a) and (b) in such a way to inclnde all reciproeal system and to derive again stability properties of the system. At the end a connection of the results with the classification problem of differential equations is attempted.*

1. Introduction. A lot of phenomena in physics are governed by reciprocal linear ω - periodic differential equations of the form

$$\dot{x} = A(t)x, \tag{1}$$

where $x \in R^n$, R^n the n - dimensional Euclidean space, A(t) a real $n \times n$ ω - periodic matrix of period $\omega > 0$ with continuous elements. Although we are concerned only with R^n here, our results are true with only minor changes when R^n is replaced by an arbitrary Banach space. If X(t) is the principal fundamental matrix of (1), then according to the theory of Floquet

$$X(t) = P(t)\exp(Bt), \tag{2}$$

where P(t) is an $n \times n$ ω - periodic matrix and B a $n \times n$ constant matrix. The characteristic roots of X (ω) are called characteristic multipliers of (1). The equation (1) is said to be reciprocal if ρ a characteristic multiplier of (1) implies ρ^{-1} is also a characteristic multiplier of (1).

Next proposition [2, p. 132] is only a sufficient condition for the reciprocality of the equation (1). Let A^T denote the transpose of any matrix A, I the identity matrix and 0 the zero matrix.

Proposition 1. If there exists a nonsingular $n\times n$ matrix D such that

$$\text{(i)}\ \ DA(t) + A^T(t)D = 0 \ \text{ or } \ \text{(ii)}\ \ DA(t) + A(-t)D = 0,$$

then equation (1) is reciprocal. Moreover, the principal fundamental matrix X(t) of (1) satisfies

$$\text{(i}'\text{)}\ \ X^T(t)DX(t) = D \qquad \text{or} \qquad \text{(ii}'\text{)}X^{-1}(-t)DX(t) = D.$$

2. D a constant matrix. In this paragraph we obtain stability properties of (1) coming from (i′) of Proposition 1, and we give an equivalent proposition of (1).

In the following, stability of (1) is meant Liapunov stability over the whole interval $(-\infty, \infty)$, which is equivalent to the boundedness of all solutions of (1) in $(-\infty, \infty)$. At first we distinguish the following cases for the matrix D:

a) Suppose that $D = I$. Then, from (i′)

$$X^T(t)\,X(t) = I.$$

Hence, the ranges of the solutions are on spherical surfaces for all $t\in(-\infty, \infty)$ and therefore (1) is stable.

b) Suppose that $D = \text{diag}\,(d_1, d_2,\dots, d_n)$, $d_i \neq 0$, $i = 1, 2,\dots, n$. Then, if $X(t) = (x_{ij}(t))$, we get, from (i′),

$$\sum_{i=1}^{n} d_i x_{ij}^2(t) = d_j \qquad j = 1, 2,\dots, n.$$

This means that, if all d_j $j = 1, 2,\dots, n$ are of the same sign, equation (1) is stable, otherwise unstable.

c) Suppose that $D + D^T = \Delta$ is a nonsingular matrix. Then a similar relation of (i) holds with $\Delta = (\delta_{ij})$ instead of D, which is now a symmetric matrix. Therefore, if we compare the diagonal elements of (i) we take

$$\varphi_i^T(t)\,\Delta\varphi_i(t) = \delta_{ii}, \tag{3}$$

where $\varphi_i(t) = (x_{1i}(t), x_{2i}(t),\dots, x_{ni}(t))^T$, $i = 1, 3, \dots, n$, the n columns of X(t). By the same argument as before, if the quadradic form (3), which is obviously a first intergal of (1), is definite, then (1) is stable, otherwise unstable.

d) The restriction of Δ being a nonsingular matrix in case (c) is not necessary. In fact, the quadradic form (3) will be definite, indefinite or semidefinite in general. If it is definite, we have stability because the ranges of the solutions of (1) are in hyperellipsoidal surfaces, otherwise we have instability.

Let now the $n \times n$ matrices $A = (a_{ij}(t))$ and $D = (d_{ij})$ and the $n^2 \times n^2$ block matrices B_1, B_2, where

$$B_1(t) = (a_{ik}(t)I)^T + \text{diag}\,(A^T(t),..., A^T(t)),$$

$$B_2(t) = (a_{ik}(-t)I)^T + \text{diag}\,(A^T(t),..., A^T(t)).$$

Consider the homogeneous systems

$$B_1(t)x = 0 \qquad (4a)$$

or

$$B_2(t)x = 0. \qquad (4b)$$

Proposition 2. Proposition 1 is equivalent to: (a) The rank of B_1 or B_2 is less than n^2. (b) (4a) or (4b) have independent of t nontrivial solutions and (c) The matrix D, which is composed from the previous n^2 - dimensional vector solution of (4a) or (4b) if the first n coordinates of it constitute the first row, the n next the second row and so on, is nonsingular.

Proof. To prove that (a), (b), (c) imply Proposition 1 we argue as follows. If (a) and (b) are fulfiled then (4a) or (4b) has a nontrivial constant solution. If we construct from that solution the $n \times n$ matrix D it will be nonsingular, according to (c). Rewriting the first members of (4a) or (4b) properly we find (i) or (ii) of Proposition 1. Conversely, suppose that Proposition 1 is true. Then, if all the elements of matrix D are numbered in the order of the rows of the matrix, the resulting n^2 - dimensional vector satisfies (4a) or (4b). But D is a nonsingular constant matrix, so (4a) or (4b) has a nontrivial solution. Therefore, (a), (b) and (c) are satisfied and the proof is complete.

The following examples are applications of the Proposition 2.

Example 1. Let A(t) in equation (1) is a 2×2 ω - periodic matrix. At first, in order (a) is valid, $\det B_1(t) = 0$, which implies

$$[\alpha_{11}(t) + \alpha_{22}(t)]^2\ [\alpha_{11}(t)\alpha_{22}(t) - \alpha_{12}(t)\alpha_{21}(t)] = 0.$$

Suppose that $\alpha_{11}(t) = \alpha_{22}(t)$. Solving the corresponding system (4a), we find

$$D = \begin{bmatrix} -\mu \dfrac{\alpha_{21}(t)}{\alpha_{12}(t)} & 2\mu \dfrac{\alpha_{11}(t)}{\alpha_{12}(t)} - \lambda \\ \lambda & \mu \end{bmatrix},$$

where λ, μ arbitrary constants. Therefore, condition (b) is satisfied if $\alpha_{21}(t) : \alpha_{12}(t) = \rho$, $\alpha_{11}(t) : \alpha_{12}(t) = \nu$ are independent of t and condition (c) if det $|D| \neq 0$. Hence, we find the reciprocal system

(a) $$\dot{x} = \alpha(t) \begin{bmatrix} \nu & 1 \\ \rho & -\nu \end{bmatrix} x.$$

In particular, if A is a 2×2 constant matrix, then we find that (1) is reciprocal if it has the special form

(a′) $$\dot{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix} x.$$

Example 2. Consider now a 2×2 matrix A(t) and suppose that det $B_2(t) = 0$. Then after some algebraic calculations we find

$$\det B_2(t) = [\alpha_{11}(t)+\alpha_{11}(-t)]\ [\alpha_{11}(-t)+\alpha_{22}(t)]\ [\alpha_{11}(t)+\alpha_{22}(-t][\alpha_{22}(t)+\alpha_{22}(-t)]+$$

$$+[\alpha_{12}(t)\alpha_{21}(t)-\alpha_{12}(-t)\alpha_{21}(-t)]^2-\alpha_{12}(t)\alpha_{21}(t)\ \big[[\alpha_{11}(-t)+\alpha_{11}(t)]\ [\alpha_{11}(-t)+\alpha_{22}(t)]+$$

$$+[\alpha_{11}(t)+\alpha_{22}(-t)]\ [\alpha_{22}(t)+\alpha_{22}(-t)]\big]-\alpha_{12}(-t)\alpha_{21}(-t)\ \big[[\alpha_{11}(t)+\alpha_{11}(-t)]\ [\alpha_{11}(t)+$$

$$+\alpha_{22}(-t)]+[\alpha_{11}(-t)+\alpha_{22}(t)]\ [\alpha_{22}(t)+\alpha_{22}(-t))\big].$$

We distringuish two cases: 2_1). Case $a_{12}(t) = a_{12}(-t) = 0$ and $a_{11}(t)+a_{11}(-t) = 0$. Then, from condition (a) investigating the corresponding homogenous system we get that conditions (b) and (c) are satisfied, if

$$\alpha_{21}(t) = \alpha_{21}(-t),\ \alpha_{11}(-t) + \alpha_{22}(t) \neq 0,\ \alpha_{22}(t) + \alpha_{22}(-t) = 0.$$

Hence, we find the reciprocal system

(b) $$\dot{x} = A(t)x = \begin{bmatrix} \alpha_{11}(t) & 0 \\ 0 & \alpha_{22}(t) \end{bmatrix} x; \quad A(t) = -A(-t)$$

and

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}, \qquad d_{11},\ d_{22} \neq 0.$$

If furthermore $\alpha_{11}(-t) + \alpha_{22}(t) = 0$, then $\alpha_{11}(t) = \alpha_{22}(t)$ and concequently

(b′) $$\dot{x} = \begin{bmatrix} \alpha_{11}(t) & 0 \\ 0 & \alpha_{11}(t) \end{bmatrix} x; \qquad D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \quad \det D \neq 0.$$

2_2) Case $\alpha_{12}(t) = \alpha_{21}(-t) = 0$ and $\alpha_{11}(-t)+\alpha_{22}(t) = 0$. Then, if $\alpha_{21}(t) \neq 0$, conditions (a), (b) and (c) are satisfied if $\alpha_{11}(t)+\alpha_{11}(-t) = \lambda\alpha_{21}(t)$, λ a constant and $\alpha_{12}(t) = \alpha_{21}(-t)$, when

(c) $$\dot{x} = \begin{bmatrix} \alpha_{11}(t) & 0 \\ \alpha_{21}(t) & -\alpha_{11}(-t) \end{bmatrix} x; \qquad D = \begin{bmatrix} -1 & \lambda \\ 1 & 1 \end{bmatrix}, \quad \lambda \neq -1.$$

All above systems (a), (a′), (b), (b′) and (c) are explicitly solvable and we find directly that in each of them $X(\omega)$ has reciprocal roots. In particular the last one is reciprocal because $\alpha_{11}(t)$-$\alpha_{11}(-t)$ is an odd function of t.

Remark 1. From the Examples 1 and 2 we show that (a) - (c) are reciprocals systems, nevertheless there exists the system

$$\dot{x} = \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) \\ \nu a_{11}(t) & \nu a_{12}(t) \end{bmatrix} x$$

which you can prove that it is reciprocal, if $\int_0^\omega [\alpha_{11}(u) + \nu\alpha_{12}(u)]du=0$, but which is not coming from conditions (i) and (ii) of Proposition 1.

3. D a matrix function of t. From Proposition 2 and even from above examples we show that the elements of the matrix A(t) must be connected with very restrictive conditions for being reciprocal the corresponding system. Concequently Proposition 1 includes very few elements of the reciprocals systems. Therefore, the necessity of generalizing Proposition 1 is obvious and that is what we are going to do in the following proposition.

Proposition 3. The linear periodic system (1) is reciprocal if and only if

(I) $\dot{D}(t) - A(t)D(t) - D(t)A^T(t) = 0$ or (II) $\dot{D}(t)-A(t)D(t)-D(t)A(-t) = 0$,

has an ω - periodic solution D(t) such that D(o) is nonsingular.

Proof. The condition is sufficient. Consider the representation of the principal fundamental matrix in the form of Floquet (2) and C a nonsingular $n\times n$ matrix. Then

$$D(t) = X(t)\, CX^T(t) \tag{5a}$$

or

$$D(t) = X(t)\, CX^{-1}(-t), \tag{5b}$$

are solutions of (I) and (II) correspondingly. In fact,

$$\dot{D}(t) = \dot{X}(t)\, CX^T(t) + X(t)\, C\dot{X}^T(t) =$$

$$= A(t)X(t)CX^T(t) + X(t)CX^T(t)A^T(t) =$$

$$= A(t)\, D(t) + D(t)\, A^T(t)$$

and, by virtue of the identity for the fundamental matrix of (1) [1,p. 37] $(X^{-1})^{\cdot} = -X^{-1}\dot{X}X^{-1}$,

$$\dot{D}(t) = \dot{X}(t)CX^{-1}(-t) + X(t)\, C\frac{d}{du}X^{-1}(u)\frac{du}{dt} =$$

$$= A(t)X(t)CX^{-1}(-t) + X(t)C[-X^{-1}(u)\dot{X}(u)X^{-1}(u)](-1) =$$

$$= A(t)X(t)CX^{-1}(t) + X(t)CX^{-1}(u)A(u)X(u)X^{-1}(u) =$$

$$= A(t)D(t) + D(t)A(-t)$$

Setting $t = 0$, we find $C = D(o)$ and taking into consideration (2),

$$D(\omega) = e^{B\omega}D(o)e^{B^T\omega} = D(o) \tag{6a}$$

or

$$D(\omega) = e^{B\omega}D(o)e^{B\omega} = D(o)\,, \tag{6b}$$

Hence, $X(\omega)$ is similar to $X^{-1}(\omega)$ in both cases.

Conversely if (1) is reciprocal then $X(\omega)$ is similar to $X^{-1}(\omega)$. Therefore there exists a constant nonsingular matrix D_0 such that (6a) or (6b) are satisfied with D_0 instead of $D(o)$. But

$$D(t) = X(t)D_0X^T(t) \qquad \text{or} \qquad D(t) = X(t)D_0X^{-1}(-t)$$

is an ω - periodic solution of (I) or (II) such that $D(o)$ is nonsingular In fact,

$$D(t+\omega) = P(t+\omega)e^{B(t+\omega)}D_0e^{B^T(t+\omega)}P^T(t+\omega) = P(t)e^{Bt}D_0e^{B^Tt}P^T(t) = D(t),$$

or

$$D(t+\omega) = P(t+\omega)e^{B(t+\omega)}D_0e^{B(t+\omega)}P^{-1}(-t-\omega) = P(t)e^{Bt}D_0e^{Bt}P^{-1}(-t) = D(t).$$

Moreover, $D(o) = D_0$ is nonsingular in both cases.

Remark 1. From (5a) we find that the symmetric matrix $\Delta(t) = D(t) + D^T(t)$ is also a solution of (I). Therefore we have the following quadradic form

$$\delta_{ii}(t) = (x_{i1},\ldots,x_{in})\ \Delta(o)\ (x_{i1},\ldots,x_{in})^T.$$

Concequently, if $\Delta(o)$ is a definite matrix, (1) is stable over $(-\infty, \infty)$ because all the coordinates of any solution are bounded. Otherwise (1) is unstable.

Remark 2. It is obvious now that Proposition 1 in [2, p. 132] is a special case of our Proposition 3, corresponding to an independent of t nonsingular solution of (I) or (II) and in that case (i) or (ii) of Proposition 1 holds with D^{-1} instead of D.

Remark 3. Conditions (I) and (II) of Proposition 3 can be replaced by

$$(I')\ \dot{D}(t)+D(t)A(t)+A^T(t)D(t) = 0 \ \text{or}\ (II')\ \dot{D}(t)+D(t)A(t)+A(-t)D(t)=0$$

and now conditions (i) and (ii) of Proposition 1 correspond to any nonsingular constant solution of (I′) or (II′). Note that the solutions of (I′) and (II′) for any nonsingular matrix C, are given by

$$D(t) = [X(t)CX^T(t)]^{-1} \ \text{or}\ D(t) = [X(t)CX^{-1}(-t)]^{-1}.$$

Actually,

$$\dot{D}(t) = -[X(t)CX^T(t)]^{-1} [\dot{X}(t)CX^T(t) + X(t)C\dot{X}^T(t)] [X(t)CX^T(t)]^{-1} =$$

$$= -[X(t)CX^T(t)]A(t)X(t)CX^T(t)[X^T(t)]^{-1}C^{-1}X^{-1}(t) -$$

$$-[X^T(t)]^{-1}C^{-1}X^{-1}(t)X(t)CX^T(t)A^T(t)[X(t)CX^T(t)]^{-1} =$$

$$= -D(t)A(t) - A^T(t)D(t)$$

and similary

$$\dot{D}(t) = -[X(t)CX^{-1}(-t)]^{-1}\left[\dot{X}(t)CX^{-1}(-t) + X(t)C\frac{d}{du}X^{-1}(u)\frac{du}{dt}\right]$$

$$[X(t)CX^{-1}(-t)]^{-1} = -[X(t)CX^{-1}(-t)]^{-1}A(t)X(t)CX^{-1}(-t)X(-t)C^{-1}X^{-1}(t) -$$

$$-X(-t)C^{-1}X^{-1}(t)X(t)CX^{-1}(u)\dot{X}(u)X^{-1}(u)[X(t)CX^{-1}(-t)]^{-1} =$$

$$= -D(t)A(t) - A(t)D(t).$$

Finally we point out the connection of our study with the known classification problem of differential equations. The ω - periodic linear differential systems

$$\dot{x} = A(t)x \tag{7a}$$

$$\dot{y} = B(t)y \tag{7b}$$

are called ω - equivalent, if there exists an ω-periodic nonsingular continuously differentiable matrix S(t) such that the substitution

$$x(t) = S(t)y(t) \tag{8}$$

transforms (7a) to (7b).

It is well known [2], that (7a) and (7b) are ω - equivalent if and only if

$$\dot{S}(t) - A(t)(S(t) + S(t)B(t) = 0$$

has an ω - periodic solution. Therefore comparing it with (I) or (II) of Proposition 3, we find that (1) is reciprocal if and only if (1) is ω - equivalent to its adjoint equation $\dot{y} = -A^T(t)y$ or if and only if (1) is ω - equivalent to $\dot{y} = -A(-t)y$.

Next Proposition can be taken as an equivalent definition of ω-equivalent equations.

Proposition 4. (7a) and (7b) are ω - equivalent if and only if they have the same characteristic multipliers with the same nullity.

Proof. Let X(t) be the principal fundamental matrix of (7a). Then from (8), the principal fundamental matrix of (7b) is $Y(t) = S^{-1}(t)X(t)S(o)$ which, for $t = \omega$, implies $Y(\omega) = S^{-1}(\omega)X(\omega)S(o)=S^{-1}(o)X(\omega)S(o)$. This means that $Y(\omega)$ is similar to $X(\omega)$. The converse is obvious.

On the other hand if P(t) and B are the matrices in the formula of Floquet (2), the substitution $x(t) = P(t)y(t)$ transforms (1) to $\dot{y} = By$. Therefore to find the ω - equivalence classes of ω - periodic equations we start with the canonical forms of Jordan and we apply the transformation (8). Finally to find the ω - equivalence classes of reciprocal systems we start with matrices in the form of Jordan with opposite characteristic roots and we apply (8).

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ΠΕΡΙΛΗΨΙΣ

ΣΥΝΘΗΚΑΙ ΙΝΑ ΕΝ ω - ΠΕΡΙΟΔΙΚΟΝ ΣΥΣΤΗΜΑ ΓΡΑΜΜΙΚΩΝ ΔΙΟΦΟΡΙΚΩΝ ΕΞΙΣΩΣΕΩΝ ΕΙΝΑΙ ΑΝΤΙΣΤΡΟΦΟΝ

Ὑπὸ

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Ἕν σύστημα ω - περιοδικῶν γραμμικῶν διαφορικῶν ἐξισώσεων $\dot{x} = A(t)x$ εἶναι ἀντίστροφον ἐὰν οἱ χαρακτηριστικοί του πολλαπλασιασταὶ εἶναι ἀντίστροφοι. Εἶναι γνωστὸν [2, p. 132] ὅτι ἱκαναὶ συνθῆκαι πρὸς τοῦτο εἶναι ἡ ὕπαρξις μὴ ἰδιάζοντος σταθεροῦ πίνακος D τοιούτου ὥστε:

(a) $DA(t)+A^T(t)D = 0$ ἤ b) $DA(t)+A(-t)D=0$

Εἰς τὴν παροῦσαν ἐργασίαν, 1ον) Δίδομεν τρόπον εὑρέσεως τοῦ πίνακος D (ὅταν οὗτος ὑπάρχῃ) ὥστε νὰ ἰσχύουν αἱ συνθῆκαι (a) καὶ (b) καὶ βάσει αὐτοῦ κατασκευάζομεν παραδείγματα ἀντιστρόφων συστημάτων. Ἐπὶ πλέον, ἐξάγομεν συμπεράσματα ὡς πρὸς τὴν εὐστάθειαν τοῦ συστήματος διὰ τῆς μελέτης τῆς συνθήκης (a). 2ον) Γενικεύομεν τὰς συνθήκας (a) καὶ (b) καὶ μάλιστα κατὰ τρόπον ὥστε αὗται νὰ εἶναι ἱκαναὶ καὶ ἀναγκαῖαι ὡς ἑξῆς: Ἵνα τὸ ἄνω τέρω σύστημα εἶναι ἀντίστροφον πρέπει καὶ ἀρκεῖ νὰ ὑπάρχῃ μὴ ἰδιάζων ω - περιοδικὸς πίναξ D(t) τοιοῦτος ὥστε:

(a′) $\dot{D}(t)-A(t)D(t)-D(t)A^T(t) = 0$ ἤ (b′) $\dot{D}((t)-A(t)D(t)-D(t)A(-t) = 0$.

Ὁμοίως, διὰ τῆς μελέτης τῆς συνθήκης (a′) ἐξάγομεν συμπεράσματα ὡς πρὸς τὴν εὐστάθειαν τοῦ συστήματος. 3ον) Τελικῶς, ἐπιτυγχάνομεν μίαν ταξινόμησιν τῶν ω - περιοδικῶν συστημάτων καὶ εἰδικῶς τῶν ω - περιοδικῶν ἀντιστρόφων συστημάτων.