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ON THE BOUNDS OF THE TRAJECTORIES OF DIFFERENTIAL SYSTEMS WITH PERTURBATIONS

By

THOMAS KIVENTIDIS

Department of Mathematics, University of Thessaloniki

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Abstract: *The behaviour of differential systems with perturbations is investigated with respect to certain time - varying subsets of the state space and their properties do not only yield information about the stability of a system but also estimates of the bounds of the system trajectories.*

The results which are established yield suffieient conditions for stability and involve the existence of Liapunov— like functions which do not appear to possess the usal definiteness requirement on V and $\dot{V}$.

1. NOTATIONS AND DEFINITIONS

Consider the differential equations

$$\dot{x} = f(t,x) \qquad (1')$$

$$\dot{x} = f(t,x) + u(t,x) \qquad (1)$$

where $f, f+u \in C([t_0,\infty) \times R^n, R^n)$, and we suppose that we have uniqueness and continuity of solutions with respect to initial data.

We denote with $\bar{S}(t)$ the closure of $S(t)$, with $cS(t)$the complement of $S(t)$ and with $\partial S(t)$ denote the boundary of $S(t) \subset R^n$.

Also, the function $V:[t_0, \infty) \times R^n \to R$ is of the class C^1 with respect to (t,x), ∇V is the gradient of V, $\dot{V}_f(t,x) = \frac{\partial V}{\partial t} + \langle \nabla V, f \rangle$, $\langle \cdot \rangle$ where denote the inner product, and

$$V_m^{\partial s(t)}(t) = \min_{x \in \partial s(t)} V(t,x), \qquad V_M^{\partial s(t)}(t) = \max_{x \in \partial s(t)} V(t, x).$$

Let $S(t) \subset R^n$, $\forall\, t \in [t_0, \infty)$ and we assume that:

(i) (S(t) is an open region which is simply connected,
(ii) S(t) is bounded and cS(t) is connected set,
(iii) $\lim_{t \to t'} S(t) = S(t')$, for all $t', t \in [t_0, \infty)$.

DEFINITION 1. The system (1) is stable with respect to $(S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, $S_0 = \bar{S}_0(t_0) \subset S(t_0)$, $\partial S_0 \cap \partial S(t_0) = \varnothing$, if for every trajectory x(t), the conditions $x(t_0) \in S_0(t_0)$ and $\| u(t,x) \| \leq \delta$, $\forall\, t \geq t_0$, $x \in S(t)$ implies that $x(t) \in S(t)$, for all $t \geq t^0$.

DEFINITION 2. The system (1) is uniformly stable with respect to $(S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, $\bar{S}^0(t) = S_0(t) \subset S(t)$, $\partial S_0(t) \cap \partial S(t) = \varnothing$, $\forall\, t \geq t_0$, if for every trajectory x(t), the conditions $x(t_1) \in S_0(t_1)$ and $\| u(t,x) \| \leq \delta$ $\forall\, t \geq t_1$, $x \in S(t)$, implies that $x(t) \in S(t)$ for all $t \geq t_1 \geq t_0$.

2. STABILITY THEOREM

The theorem is based in the following lemma.

LEMMA: Suppose $S: [t_0, \infty) \to P(R^n)$, $\lim_{t \to t'} S(t) = S(t')$ — $P(R^n)$ is the set of all subsets of R^n-and such that S(t) is bounded and cS(t) is connected for all t', $t \in [t_0, \infty)$. Then the system (1) is stable with respect to $(S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, if and only if the system is stable with respect to $(\partial S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, where $S_0(t_0)$ is closed subsed of $S(t_0)$.

The proof of the lemma is similar to the proof of the Theorem 3 of J. Heinen [2].

THEOREM. The system (1) is stable with respect to $(S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, where $\bar{S}_0(t_0) = S_0(t_0) \subset S(t_0)$ and $\partial S_0(t_0) \cap \partial S(t_0) = \varnothing$, if there exists a continuously differentiable function V(t,x) and two realvalued functions $\varphi(t)$, $\rho(t)$ which are integrable over $[t_0, \infty)$ such that

(i) $\| \nabla V \| \leq \rho(t)$, $\forall\, t \geq t_0$, $x \in S(t)$,
(ii) $\dot{V}_f(t,x) < \varphi(t)$, $\forall\, t \geq t_0$, $x \in S(t)$,
(iii) $\int_{t_0}^{t} [\varphi(s) + \delta\rho(s)]ds \leq V_m^{\partial s(t)}(t) - V_M^{\partial s_0(t_0)}(t_0)$, $\forall\, t > t_0$.

PROOF.

Let x(t) denote an arbitrary trajectory of (1) with initial conditions such that $x(t_0) \in \partial S_0(t_0)$. Assume that there exists a $t_1 \in (t_0, \infty)$, the first point in (t_0, ∞) such that $x(t_1) \in \partial S(t_1)$.

Evidently for all $t \in [t_0, t_1)$, $x(t) \in S(t)$.

Now we can write

$$V(t_1, x(t_1)) = V(t_0, x(t_0)) + \int_{t_0}^{t_1} \dot{V}(s, x(s))ds.$$

In view of hypothesis (i) and (ii), one obtains

$$V(t_1, x(t_1)) < V_M^{\partial S_0(t_0)}(t_0) + \int_{t_0}^{t_1} [\varphi(s) + \delta\rho(s)]ds$$

and by applying hypothesis (iii) to the above inequality it follows that

$$V(t_1, x(t_1)) < \partial_M^{\partial S_0(t_0)}(t_0) + V_m^{\partial S(t_1)}(t_1) - V_M^{\partial S_0(t_0)}(t_0),$$

or $$V(t_1, x(t_1)) < V_m^{\partial S(t_1)}(t_1)$$

which is a contradiction to the original assumption.

Hence the system (1) is stable with respect to $(\partial S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, and by lemma, is stable with respect to $(S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$.

REMARKS

a) If, in Theorem, the hypothesis (iii) to become

$$\int_{t_1}^{t} [\varphi(s) + \delta\rho(s)]ds \leq V_m^{\partial S(t)}(t) - V_M^{\partial S_0(t_1)}(t_1), \ \forall\ t > t_1 \geq t_0$$

then the system (1) is uniformly stable with respect to $(S_0(t_0), S(t), \delta, t_0, \|\cdot\|)$, $S_0(t_1) = \overline{S}_0(t_1) \subset S(t_1)$ and $\partial S_0(t_1) \cap \partial S(t_1) = \varnothing$, $\forall\ t_1 \geq t_0$.

b) If we will suppose more that $\lim_{t \to \infty} S(t) = \{a\}$, $a \in R^n$, then the system (1) is asymptotically stable (resp. uniformly asymptotically) with respect to $(S_0(t_0), S(t), \{a\}, \delta, t_0, \|\cdot\|)$ i.e. is stable (resp. uniformly stable) with respect to ($S_0(t_0)$, $S(t)$, δ, t_0, $\| . \|$) and for every solution $x(t)$, of the system (1), with $x(t_0) \in S_0(t_0)$, we have $\lim_{t \to \infty} x(t) = a$.

c) For special case we can to obtains $S_0(t) = S_0 = \{x: \| x \| \leq \varepsilon_1\}$ and $S(t) = S = \{x: \| x \| < \varepsilon_2\}$, with $\varepsilon_1 < \varepsilon_2$.See[4].

d) This type of stability have also been discussed in [3] for the case $u(t,x) \equiv 0$ i.e. for the unperturbed system $\dot{x} = f(t, x)$.

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ΠΕΡΙΛΗΨΗ

ΠΕΡΙ ΤΩΝ ΦΡΑΓΜΕΝΩΝ ΛΥΣΕΩΝ ΔΙΑΦΟΡΙΚΩΝ ΣΥΣΤΗΜΑΤΩΝ ΜΕ ΔΙΑΤΑΡΑΞΕΙΣ

Ὑπὸ

ΘΩΜΑ ΚΥΒΕΝΤΙΔΗ

(Μαθηματικό Σπουδαστήριο Παν/μίου Θεσ/νίκης)

Μελετᾶται ἡ συμπεριφορὰ διαφορικῶν συστημάτων μὲ διαταράξεις ὡς πρὸς μερικὰ ὑποσύνολα τοῦ χώρου φάσεως ποὺ μεταβάλλονται μὲ τὸ χρόνο καὶ οἱ ἰδιότητές-τους δὲν μᾶς δίνουν μόνο πληροφορίες γιὰ τὴν εὐστάθεια τοῦ συστήματος, ἀλλὰ ἐπίσης καὶ γιὰ τὰ φράγματα τῶν λύσεών-του.

Τὰ ἀποτελέσματα ποὺ ἀναφέρονται δίνουν ἱκανὲς συνθῆκες γιὰ τὴν εὐστάθεια καὶ χρησιμοποιοῦν τὴν ὕπαρξη μιᾶς Liapunov συνάρτησης V, χωρὶς τὴν ἀπαίτηση τοῦ συνηθισμένου θετικὰ ἢ ἀρνητικὰ ὁρισμένη, γιὰ τὰ V, $\dot{V}$.