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EXACT FADDEEV EQUATIONS OF A THREE-BODY STATE IN THE SU(3) AND MOMENTUM REPRESENTATION

by

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Abstract: *The Faddeev equations are derived for the wave amplitudes of the bound-state of three particles in the S-state, in the case when the masses of two of the particles are equal, by neglecting the spins. The wave amplitudes are classified according to the SU (3) and momentum representation.*

1. INTRODUCTION

The object of this paper is to give the homogeneous and once-iterated exact Faddeev integral equations [1] for three particles when the masses of two of them are equal by neglecting the spins, which are derived, to the best of our knowledge, for the first time in the SU (3) representation. We derive the new equations which in operator formalism are not obtained by simply taking $m_1=m_2$ in the standard Faddeev equations (for $m_1\neq m_2\neq m_3$) as one might have expected, but through a procedure, the details of which will be given in a forthcoming paper [2], by using the results of reference [3].

The spatial wave amplitudes for the present system in momentum variables are classified according [3] to the components of the symmetric SU (3) state. This type of symmetry of components is used for the classification, in order that the general statement of the Pauli principle holds when it is applied to the three-body wave amplitudes. The latter are symmetric, when the particles 1,2 exchange their coordinates in momentum space at the centre of mass system.

The Faddeev kernel in this represtation has only one integrating variable, the three-body momentum $|\underline{k}|$. The SU(3) harmonics have been derived by Dragt [4]. They were used first by Hochberg, Lee, and Sibbel [3] and then, by Sibbel [5] and by Saloupis [6] in their calculations for the ^3H and ^3He, after a modification was made [3] in which the SU (3) polynomials became useful in practice.

2. THE COORDINATES USED

Let $\underline{k}_1$, $\underline{k}_2$, $\underline{k}_3$ be the vectors in momentum space for the particles 1, 2 and 3. We use, instead, the vectors η_k, ξ_k, $k^{(3)}$ obtained by transforming the $\underline{k}_i$, $\underline{k}_j$, $\underline{k}_k$ ($i, j, k=1, 2, 3$; $i \neq j \neq k$) with the help of the matrix[4]

$$\begin{bmatrix} -\dfrac{c\mu}{d_k m_i} & \dfrac{c\mu}{d_k m_j} & 0 \\ -\dfrac{c\mu d_k}{m_i+m_j} & -\dfrac{c\mu d_k}{m_i+m_j} & \dfrac{c\mu d_k}{m_k} \\ \dfrac{1}{c^2} & \dfrac{1}{c^2} & \dfrac{1}{c^2} \end{bmatrix}.$$

The parameters c, μ, d_k are given as follows:

$$c = 3^{1/4}, \quad \mu = \sqrt{\frac{m_1 m_2 m_3}{M}}, \quad M = m_1+m_2+m_3 \text{ and } d_k = \sqrt{\mu\frac{m_i+m_j}{m_i m_j}}.$$

The three particles are treated symmetrically in $\underline{\eta}_k$, $\underline{\xi}_k$, $\underline{k}^{(3)}=0$ (i.e. at the centre of mass system). $\underline{\eta}_k$ is the relative momentum vector of the particles i, j. $\underline{\xi}_k$ is the momentum vector for the k-th particle.

The kinetic energy T is now given by

$$T = \hbar^2 \left(\frac{k^2_1}{2m_1} + \frac{k^2_2}{2m_2} + \frac{k^2_3}{2m_3}\right) = h^2 \left(\frac{\eta^2+\xi^2}{2c^2\mu}\right) + \frac{h^2c^4}{2M} k^{(3)2}$$

(MeV). With $\eta^2+\xi^2=k^2=\eta_i^2+\xi_i^2$ for i=1,2,3, the energy becomes,

$$E = \left(\frac{2c^2\mu}{\hbar^2}\right) T = k^2 = w_1+w_2+w_3 \ (fm^{-2})$$ with the convention that

$\frac{\hbar^2}{2c^2\mu} = 1$. The w_i are defined by $w_i = (\frac{c^2\mu}{m_i}) k^2_i$ fm^{-2}. The η^2_k, ξ^2_k are now given by

$$\eta^2_k = (\frac{m_k}{m_i+m_j}) [(\frac{m_i+m_j}{m_k}) w_i + (\frac{m_i+m_j}{m_k}) w_j - w_k],$$

$$\xi^2_k = \frac{M}{m_i+m_j} w_k. \qquad (1)$$

It should be noted that we write for our convenience the equation for η_k as $\eta_k = \eta_k(w_k w_i w_j)$.

3. THE DERIVED EXACT EQUATIONS

The Faddeev equations for the wave function $\bar{\psi}$ of a three-body bound state are homogeneous and are given [1] as follows:

$$\psi^i = -G_0(z)\ T_i(z)\ (\psi^j + \psi^k), \qquad (2)$$

$i,j,k=1,2,3$; $i \neq j \neq k$, with $\bar{\psi} = \sum_{i=1}^{3} \psi^i \cdot z =$ the total three-body energy; $G_0(z) = \frac{1}{H_0 - z}$, the Green's function operator and H_0 is the Hamiltonian of three free particles. The operator T is expressed as follows:

$\langle \underline{k} | T_i(z) | \underline{k}' \rangle = \delta(\underline{\xi}_i - \underline{\xi}_i') \langle \underline{\eta}_i | t_i(z - \xi_i'^2) | \underline{\eta}'_i \rangle$, where $\langle \underline{k} | = \langle \underline{\xi}_i, \underline{\eta}_i |$ and so on. After a decomposition of $\langle \underline{\eta}_i | t_i(z - \underline{\xi}'^2) | \underline{\eta}_i' \rangle$ in partial waves is made the $t_i^l(\eta_i, \eta'_i; z - \xi_i'^2)$, for $l=0$, satisfies the Lippmann-Schwinger equation [3]

$$t_i^0(\eta_i, \eta_i'; z - \xi_i'^2) =$$

$$V_i^0(\eta_i, \eta_i') - 2\pi \int_0^\infty \frac{\eta_i'' V_i^0(\eta_i, \eta_i'') t_i^0(\eta_i'', \eta_i'; z - \xi_i'^2) d\eta_i''^2}{\eta_i''^2 - (z - \xi_i'^2)}.$$

The $V_i^0(\eta_i, \eta_i')$ is the nuclear potential $V_i^0(r_i)$ in momentum space.

Equations (2) in the SU(3) and the momentum variables (λ, ν and k^2, respectively) take [3], after one iteration, the following form in which they are still coupled in the three terms ψ^i: $\langle k^2,\lambda\nu|\psi^i\rangle =$

$$\sum_{j=1}^{3} \sum_{\lambda',\nu'} \int_0^\infty \langle k^2,\lambda\nu\,|\,K^{i,j}(z)\,|\,k'^2,\lambda'\nu'\rangle dk'^2 \langle k'^2,\lambda'\nu'\,|\,\psi^j\rangle, \qquad i=1,2,3. \tag{3}$$

$K^{i,j}(z)$ has the form

$$K^{i,j}(z) = \sum_{k\neq i,j} G_0(z)T_i(z)\ \ G_0(z)T_k(z), \quad i,j,k=1,2,3. \tag{4}$$

The different terms of the r.h.s. of eq. (4) are six. We may show that eqs., (3) anc (4) reduce for the ground state, after we consider $m_1=m_2=m$, to the following exact new equations for the full wave function $\bar{\psi}$ $\langle k^2, \lambda\mu\,|\,\bar{\psi}\rangle =$

$$\frac{4}{3} \sum_{\lambda',\mu'} \int_0^\infty \langle k^2,\lambda\mu\,|\,K(z)\,|\,k'^2,\lambda'\mu'\rangle dk'^2 \langle k'^2,\lambda'\mu'\,|\,\bar{\psi}\rangle, \tag{5}$$

where the kernel is now found to be given by $K(z)=[G_0(z)\ T_1(z)G_0(z)T_2$

$$(z)]_{\Delta,\delta} + [G_0(z)T_1(z)G_0(z)T_3(z)]_{N,\mu} + [G_0(z)T_3(z)G_0(z)T_2(z)]_{N,\nu}. \tag{6}$$

The meaning of the subscripts Δ,δ, N,μ, N,ν is given at the end of this Chapter.

The Faddeev kernel in the SU(3) and momentum representation [where the SU(3) state for the particles 1,2 is symmetric to the transposition (12)] takes the form $\langle k^2, \lambda\mu\,|\,K(z)\,|\,k'^2,\lambda'\mu'\rangle=$

$$\frac{(2m+m_3)^{3/2}}{54\pi m\sqrt{m_3}} \sqrt{(\lambda+2)(\lambda'+2)}\ \ k^2k'^2 \int_0^\alpha d[\cos(2\psi^*)] \int_0^\alpha d[\cos(2\psi'')] \int_0^{2\pi} d\varphi^*$$

$$\int_0^{2\pi} d\varphi'' \langle k^2,\psi^*\varphi^*\,|\,K(z)\,|\,k'^2,\ \psi''\varphi''\rangle \cos(\mu\varphi^*)\cos^{\mu+1}(2\psi^*)\cos^{\mu'+1}(2\psi'')\cos(\mu'\varphi'')$$

$$P^{\mu,o}_{\frac{1}{2}(\frac{\lambda}{2}-\mu)}[1-2\cos^2(2\psi^*)]\ \ P^{\mu',o}_{\frac{1}{2}(\frac{\lambda'}{2}-\mu')}[1-2\cos^2(2\psi'')] \times$$

$$\times \begin{cases} 1 & \mu,\mu' \neq 0 \\ \frac{1}{2} & \mu,\mu'=0 \\ \frac{1}{\sqrt{2}} & \text{for one of } \mu,\mu'=0. \end{cases}$$

The letter α has the form

$$\alpha = \sqrt{\frac{8m_3m-2m^2_3}{3m^2+2m_3m+m^2_3}},$$

with $\alpha < 1$ for the masses ot the particles which form the $^3H_\Lambda$. The P's are the Jacobi polynomials., λ, $\lambda'=0,2,4,6,8\ldots$; μ, $\mu'=0,1,2,3,4\ldots$

The components of the symmetric SU(3) state used are characterised [3] as follows: (λ,μ), $(\lambda',\mu')=(0,0)$, $(2,1)$, $(4,2)$ $(4,0)$, $(6,1)$, $(6,3)$, $(8,4)$, $(8,2)$, $(8,0)$,...

For the case where $m_1=m_2=m_3$ eqs. (5) and (6) in the completely symmetric SU(3) state take the same form as that derived by HLS. In that case $\mu(\mu')=0,3,6,9,\ldots=\mu_3(\mu'_3)$, $\lambda(\lambda')=0,4,6,8,\ldots$, and $\alpha=1$.

The first term of K(z) on the r.h.s. of eq. (6), taken as a typical one from all of the terms, is given in the energy variables as follows:

$$[\langle k^2,\psi^*\varphi^* | G_0(z)T_1(z)G_0(z)T_2(z) | k'^2,\psi''\varphi''\rangle]_{\Delta,\delta} =$$

$$= \left[\frac{(m+m_3)^3\pi^2}{4m^2m_3}\right]_{=\Delta} \cdot \frac{1}{w^*_1+w^*_2+w^*_3-z} \cdot \frac{1}{\sqrt{w_1^* w''_3}} \cdot \tag{7}$$

$$\int_{b_1}^{b_2} \frac{t^0_1(w^*_1w^*_2w^*_3,w^*_1w''_2w_3;z-\frac{2m+m_3}{m+m_3}w^*_1)\,t^0_2(w''_2w_3w^*_1,w''_2w''_3w''_1;z-\frac{2m+m_3}{m+m_3}w''_2)\cdot dw_3}{(w^*_1+w_2''-z)+w_3}$$

where $b_1 = \left[\sqrt{\frac{m}{m_3}w^*_1} - \sqrt{\frac{m}{m_3}w_2''}\right]^2_{=\delta^-}$ and $b_2 = \left[\sqrt{\frac{m}{m_3}w^*_1} + \sqrt{\frac{m}{m_3}w_2''}\right]^2_{=\delta^+}$.

It should be noted that $\langle k^2, \psi\varphi | = \langle w_1w_2w_3 |$, as one can see from eq. (8). For remarks on the t's see the text after eq. (1). The numerator and the denominator of the integrand as well as the energy variables in the square root, are found to be, for the remaining terms of the kernel K(z) in eq. (6), as follows

$$t^0_1(w^*_1w^*_2w^*_3,w^*_1w_2w''_3;z-\frac{2m+m_3}{m+m_3}w^*_1)\,t^0_3(w''_3w^*_1w_2,w''_3w_1''w_2;z-\frac{2m+m_3}{2m}w_3'')\,dw_2,$$

$$\overset{0}{t}_3(w^*_3 w^*_1 w^*_2, w^*_3 w_1 w''_2; z - \frac{2m+m_3}{2m} w^*_3) \cdot \overset{0}{t}_2(w''_2 w^*_3 w_1, w''_2 w_3'' w_1''; z - \frac{2m+m_3}{m+m_3} w_2'')\, dw_1,$$

$$(w^*_1 + w_3'' - z) + w_2, \quad (w^*_3 + w''_2 - z) + w_1$$

and, $\sqrt{w^*_1 w^*_3}$, $\sqrt{w^*_3 w_2''}$ respectively. The Δ, δ, N, μ, N, ν used as subscripts in eq. (6) are given as follows

$$\Delta = \frac{(m+m_3)^3 \pi^2}{4m^2 m_3}, \quad N = \frac{(m+m_3)^{3/2} \pi^2}{m_3 \sqrt{2m}} \text{ and, } \delta^{\pm} = \left[\sqrt{\frac{m}{m_3} w^*_1} \pm \sqrt{\frac{m}{m_3} w''_2}\right]^2,$$

$$\mu^{\pm} = \left[\sqrt{w^*_1} \pm \sqrt{\frac{m_3}{m} w_3''}\right]^2, \quad \nu^{\pm} = \left[\sqrt{\frac{m_3}{m} w^*_3} \pm \sqrt{w_2''}\right]^2.$$

The N, and $\mu^{\pm}$, $\nu^{\pm}$ take the same place as that of Δ, and $\delta^{\pm}$ in eq. (7), respectively, chosen in accordance with the terms of K(z) to be calculated.

The w_1, w_2, w_3 are related as follows to the Dalitz-Fabri [3] coordinates φ, ψ:

$$w_1 = \frac{k^2}{3}[1 + \cos(2\psi)\cos(\varphi - \frac{2\pi}{3})], \quad w_2 = \frac{k^2}{3}[1 + \cos(2\psi)\cos(\varphi + \frac{2\pi}{3})],$$

$$\text{and} \quad w_3 = \frac{k^2}{3}[1 + \cos(2\psi) \cdot \cos\varphi]. \qquad (8)$$

The present formalism could be used in the case of spinless particles. Since, however, we are primarily interested in the case of systems like $^3H_\Lambda$ we plan to extent this formalism in order to take the spin into account.

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ΠΕΡΙΛΗΨΗ

ΑΚΡΙΒΕΙΣ ΕΞΙΣΩΣΕΙΣ FADDEEV ΜΙΑΣ ΚΑΤΑΣΤΑΣΕΩΣ ΤΡΙΩΝ ΣΩΜΑΤΩΝ ΣΤΗΝ ΑΝΑΠΑΡΑΣΤΑΣΗ ΤΟΥ ΧΩΡΟΥ ΤΩΝ ΟΡΜΩΝ ΚΑΙ ΤΗΣ ΟΜΑΔΟΣ SU(3)

ὑπὸ

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Οἱ ἐξισώσεις Faddeev γιὰ τὴν κυματοσυνάρτηση τῆς δεσμίας καταστάσεως τριῶν σωματιδίων στὴ κατάσταση S ὑπολογίζονται στὴ περίπτωση ποὺ οἱ μάζες δύο σωματιδίων εἶναι ἴσες, μὴ λαμβανομένου ὑπ' ὄψη τοῦ spin. Τὰ πλάτη στὸ χῶρο τῶν ὁρμῶν ταξινομοῦνται σύμφωνα μὲ τὴν ὁμάδα SU(3).